Problem Sheet 9

1. (4 points) Let L/K be a finite extension of non-archimedean local fields. Assume that L/K is Galois with Galois group G and ramification filtration G_s. Pick $x \in \mathcal{O}_L$ such that $\mathcal{O}_L = \mathcal{O}_K[x]$ and let $f \in \mathcal{O}_K[X]$ be its minimal polynomial. Show that

$$v_L(f'(x)) = \sum_{s=0}^{\infty}(|G_s| - 1)$$

(Remark: The ideal $(f'(x))$ is called the different of L/K. The exercise shows that it does not depend on the choice of x.)

2. (4 points) Define the following normed vector spaces:

$$\ell^\infty(\mathbb{Q}_p) = \{(a_i)_{i\in\mathbb{N}} \colon a_i \in \mathbb{Q}_p, a_i \to 0 \text{ as } i \to \infty\}, \qquad \|(a_i)\| = \sup|a_i|,$$
$$C(\mathbb{Z}_p, \mathbb{Q}_p) = \{f \colon \mathbb{Z}_p \to \mathbb{Q}_p \text{ continuous}\}, \qquad \|f\| = \sup|f(x)|.$$

Prove that there is an isomorphism of normed vector spaces $\ell^\infty(\mathbb{Q}_p) \simeq C(\mathbb{Z}_p, \mathbb{Q}_p)$.

(Hint: Observe that if $f(x) = \sum_{i=0}^{\infty} a_i \binom{x}{n}$, then a_0 can be recovered by evaluating $\Delta f(x) = f(x+1) - f(x)$ at $x = 0$. The higher a_i can be recovered similarly by applying Δ several times.)

3. (1+1+1+1 points) Recall that an object I in an abelian category $\mathcal{A}$ is *injective* if for every injection $M \hookrightarrow N$ the induced map $\mathrm{Hom}(N, I) \to \mathrm{Hom}(M, I)$ is surjective. We focus our attention on $\mathcal{A} = \mathrm{Mod}_A$, the category of modules over a ring A.

   (a) Prove that an A-module I is injective if and only if the map $\mathrm{Hom}(A, I) \to \mathrm{Hom}(\mathfrak{a}, I)$ is surjective for every ideal $\mathfrak{a} \subseteq A$.

   (b) Assume that A is a PID. Prove that an A-module I is injective if and only if for every non-zero $x \in A$, the map $I \xrightarrow{\cdot x} I$ is surjective.

   (c) Prove that every abelian group M embeds into some I such that I is injective as $\mathbb{Z}$-module.

   (d) Deduce that every module M over any ring embeds into an injective module.

(Hints: For (c), the group $I = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ satisfies the condition of (b) and is much larger than M, but there is no obvious way to embed M into I – do you see a trick? For (d), observe that there exists an adjunction $Mod_{\mathbb{Z}} \leftrightarrows Mod_A$ such that the right adjoint preserves injectives and the unit map is injective.)

4. (2+2 points) (a) Let G be any group. Consider the following complex of $\mathbb{Z}[G]$-modules:

$$\cdots \to \mathbb{Z}[G^3] \to \mathbb{Z}[G^2] \to \mathbb{Z}[G] \to \mathbb{Z} \to 0,$$

   with differentials $d \colon \mathbb{Z}[G^m] \to \mathbb{Z}[G^{m-1}]$ given by $d_m(g_1, \ldots, g_m) = \sum_{i=1}^{m}(-1)^i(g_1, \ldots, \hat{g}_i, \ldots, g_m)$. Prove that this is a projective resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$.

(b) Let G be a cyclic group of order n, fix a generator $T \in G$. Consider the following complex of $\mathbb{Z}[G]$-modules:

$$\cdots \to \mathbb{Z}[G] \to \mathbb{Z}[G] \to \mathbb{Z}[G] \to \mathbb{Z} \to 0,$$

with differential $d_m : \mathbb{Z}[G] \to \mathbb{Z}[G]$ multiplication by $T - 1$ in odd degrees and by $1 + T + \cdots + T^{n-1}$ is even degrees m. Prove that this is a projective resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$.

Please hand in your solutions in the lecture on Tuesday, 18th of December. You may work in groups of at most three students.