Problem Sheet 8

1. (1+3 points) Let $K/\mathbb{Q}_p$ be a finite extension and $\breve{K}$ be the completion of the maximal unramified extension of K.
   (a) Prove that there exists a unique automorphism φ of $\breve{K}$ fixing K such that φ reduces to the Frobenius on $\mathcal{O}_{\breve{K}}/\mathfrak{m}$.
   (b) Prove that there exist short exact sequences

$$1 \to \mathcal{O}_K^\times \to \mathcal{O}_{\breve{K}}^\times \xrightarrow{x \mapsto \phi(x)/x} \mathcal{O}_{\breve{K}}^\times \to 1,$$
$$0 \to \mathcal{O}_K \to \mathcal{O}_{\breve{K}} \xrightarrow{x \mapsto \phi(x)-x} \mathcal{O}_{\breve{K}} \to 0.$$

   (Hint: The sequences reduce to well-known sequence in Galois theory. Prove that they are exact by approximation.)

2. (4 points) Let $K/\mathbb{Q}_p$ be a finite extension. Show that any two Lubin–Tate formal group laws F_f and $F_{f'}$ become isomorphic over $\mathcal{O}_{\breve{K}}$.

3. (2+2 points) *(Weierstrass preparation theorem)* Let K be a non-archimedean local field and $\pi \in K$ be a uniformizer. Consider a power-series $f(T) = a_0 + a_1T + a_2T^2 + \cdots \in \mathcal{O}_K[[T]]$.
   (a) Assume that $a_i \in \mathfrak{m}_K$ for $i < n$, but $a_n \notin \mathfrak{m}_K$. Prove that for every $g \in \mathcal{O}_K[[T]]$ there exists a unique factorization $g = qf + r$, with $q \in \mathcal{O}_K[[T]]$ and $r \in \mathcal{O}_K[T]$ a polynomial of degree at most $n-1$.
   (b) Show that f admits a unique presentation $f = \pi^m gh$, with $m \geq 0$, g a monic polynomial satisfying $g \equiv T^n \mod \mathfrak{m}_K$ and $h \in (\mathcal{O}_K[[T]])^\times$ is a unit.

4. (4 points) Let k be a field of characteristic $p > 0$ and $F(X,Y) \in k[[X,Y]]$ a formal group law. We say that F has *height* h if

$$[p]_F = a_{p^h}X^{p^h} + a_{p^h+1}X^{p^h+1} + \ldots$$

   and $a_{p^h} \neq 0$.

   Let now $K/\mathbb{Q}_p$ be a finite extension and $F(X,Y) \in \mathcal{O}_K[[X,Y]]$ a formal group law. Let h be the height of its reduction modulo $\mathfrak{m}_K$. Prove that there exists a finite extension K'/K such that the p-torsion of the group $F(\mathfrak{m}_{K'})$ is $(\mathbb{Z}/p\,\mathbb{Z})^h$.

   (Hint: Use exercise 3.)

Please hand in your solutions in the lecture on Tuesday, 11th of December. You may work in groups of at most three students.