Problem Sheet 11

1. (a) Prove that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}[p^{-1}]/\mathbb{Z}) \simeq \mathbb{Q}_p$. Deduce that $\operatorname{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \simeq \prod_{p \text{ prime}} \mathbb{Q}_p$.

   (b) The group of finite adeles is defined to be

$$\mathbb{A}_{\mathbb{Q}}^{\infty} := \{(x_p) \in \prod_{p \text{ prime}} \mathbb{Q}_p \colon x_p \in \mathbb{Z}_p \text{ for all but finitely many } p\}.$$

   Prove that $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \simeq \mathbb{A}_{\mathbb{Q}}^{\infty}/\mathbb{Q}$.

   (Hint: For (b) use the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ and (a). For (a), prove first that $\mathbb{Q}/\mathbb{Z} \simeq \bigoplus \mathbb{Z}[p^{-1}]/\mathbb{Z}$. You will use this fact two times in the exercise.)

Some exercises recalling material we touched in the middle of term:

1. Let $K/\mathbb{Q}_p$ be a finite extension.

   - Show that the group $\mathcal{O}_K^{\times}$ is compact in the topology of the valuation.
   - Suppose L/K is an algebraic extension. Show that if the norm image $N_{L/K}(L^{\times})$ is of finite index in $K^{\times}$, it is a clopen subgroup.

2. Let R be any ring and $u \in R^{\times}$. Check that

$$F(x, y) := \frac{x + y}{1 + \frac{xy}{u}}$$

   defines a formal group law over R (not important: there is a link to velocities in physics explaining why it should...). If R contains $\mathbb{Q}$ and $u := 1$, describe explicitly an isomorphism between F and the additive formal group law $\widehat{\mathbb{G}}_a$.

3. Let F_f be the Lubin-Tate formal group law for

$$f = (1 + X)^p - 1$$

   on $\mathbb{Q}_p$ (with $L = K = \mathbb{Q}_p$).

   - Provide[1] a primitive element and compute its minimal polynomial for the field extension $\mathbb{Q}_p[\mu_{f,m}]/\mathbb{Q}_p$ for $m = 1, 2, 3$.
   - Provide an isomorphism (over $\mathcal{O}_K$) between F_f and $\widehat{\mathbb{G}}_m$.

[1] Giving its minimal polynomial is sufficient

- Compute the scalar operation $[p^2]_{f,f}$ as a power series.

4. Let F_f be the Lubin-Tate formal group law for

$$f = X^p + pX$$

on $\mathbb{Q}_p$ (with $L = K = \mathbb{Q}_p$).

- Provide[2] a primitive element and compute its minimal polynomial for the field extension $\mathbb{Q}_p[\mu_{f,m}]/\mathbb{Q}_p$ for $m = 1, 2, 3$.
- Is the formal group law F_f isomorphic (over $\mathcal{O}_K$) to $\widehat{\mathbb{G}}_m$?
- *(tricky, but well worth it!)* Let ζ be a primitive $(p-1)$-th root of unity in $\mathbb{Q}_p$ (give a detailed argument why ζ exists!). Compute the scalar operation $[\zeta]_{f,f}$ as a power series.
  [Hint: pick elements y_n such that $[p]y_n = y_{n-1}$, show that their minimal polynomials are polynomials in X^{p-1}. Conclude that there exists a Galois automorphism of the Lubin–Tate extension tower such that $\sigma y_n = \zeta y_n$ holds simultaneously for all n. Now use some theory!]
- Use the previous part of the exercise to show the following: Expanding $F_f \in \mathcal{O}_K[[X, Y]]$ as

$$F_f(X, Y) = X + Y + \sum_{i,j} g_{i,j} X^i Y^j,$$

  we have

$$g_{i,j} = 0 \qquad \text{for all indices} \qquad i + j \not\equiv 1 \bmod (p-1).$$

  (Hint: It is not necessary to attempt to compute the $g_{i,j}$ directly for this)

5. Compute

- the Galois cohomology group $H^1(\mathbb{Q}, \mu_{\overline{\mathbb{Q}},n})$, where $\mu_{\overline{\mathbb{Q}},n}$ denotes the Galois module of n-th roots of unity inside an algebraic closure $\overline{\mathbb{Q}}$.
- the Galois cohomology group $H^1(\mathbb{C}, \mathbb{Q}/\mathbb{Z})$,
- for $G := Gal(\mathbb{F}_{p^r}/\mathbb{F}_p)$ the group $H_1(G, \mathbb{Z})$,
- for $G := Gal(\mathbb{F}_{p^r}/\mathbb{F}_p)$ the Tate cohomology group $\widehat{H}^0(G, \mathbb{F}_{p^r}^\times)$.

[2] Giving its minimal polynomial is sufficient