# Problem Sheet 11

1. (2+2 points) *(A rationality criterion)*
   (a) Let $K$ be a field and $F(T) = \sum_{i=0}^{\infty} a_i T^i$ a power series. For $m, s \geq 0$, let $A_{s,m}$ be the $m+1 \times m+1$ matrix

$$A_{s,m} = \begin{pmatrix} a_s & a_{s+1} & \dots & a_{s+m} \\ a_{s+1} & a_{s+2} & \dots & a_{s+m+1} \\ \vdots & \vdots & & \vdots \\ a_{s+m} & a_{s+m+1} & \dots & a_{s+2m} \end{pmatrix}$$

   Show that $F(T)$ is a rational function if and only if there exist $M, S$ such that $\det A_{s,M} = 0$ for all $s \geq S$.

   (b) Let $F(T) = 1 + a_1 T + a_2 T^2 + \cdots \in \mathbb{Z}[[T]]$, and assume that $F$ defines a holomorphic function on some disc around 0 in $\mathbb{C}$ and that it defines an entire meromorphic function on $\mathbb{C}_p$. Prove that $F$ is the power series expansion of a rational function, i.e. the quotient of two polynomials with coefficients in $\mathbb{Q}$.

   *(Hint: For (b), try the case first where $F$ defines a function that has no poles and doesn't vanish on a big disc around 0 in $\mathbb{C}_p$; the general case uses Weierstrass preparation and is rather hard. To apply part (a) you want to estimate the determinant both p-adically and archimedically. To conclude, use that $n = 0$ is the only integer satisfying $|n|_p |n|_\infty < 1$.)*

2. (1+1+2 points) Let $G$ be a group. The *augmentation* ideal $I_G$ is the kernel of the map

$$\mathbb{Z}[G] \to \mathbb{Z} \colon \sum n_g g \mapsto \sum n_g.$$

   (a) Show that $H_0(G, M) \cong M/I_G M$ for all $G$-modules $M$.

   (b) View $\mathbb{Z}$ as a $G$-module with the trivial $G$-action. Show that $H_1(G, \mathbb{Z}) \cong I_G/I_G^2$.

   (c) Consider the map $G \to I_G/I_G^2 \colon g \mapsto g - 1 + I_G$. Show that it is a surjective homomorphism and compute its kernel.

   Finally, deduce that $H_1(G, \mathbb{Z}) \cong G^{\mathrm{ab}}$, the abelianization of $G$.

# Reflex questions

3. Give an example of a field with two discrete valuations which are not equivalent.

4. Does there exist a field $K$ with a non-trivial discrete valuation $v_1$ and a non-trivial non-discrete valuation $v_2$? Give an example or a proof of non-existence. If an example exists, could the two valuations happen to induce the same topology on $K$?

5. Let $G$ be a group (possibly non-abelian). Let $f, g : G \to \mathbb{R}^{\times}_{>0}$ be group homomorphisms. Suppose $f$ is not the trivial map (i.e. sends everything to 1) and $f(x) < 1$ holds for $x \in G$ if and only if $g(x) < 1$ holds. Prove that there exists a positive real number $\alpha$ such that
$$f(x) = g(x)^{\alpha}$$
holds for all $x \in G$.

6. Let $K$ be a complete valued field. Suppose $x \in K$ satisfies $|x| \neq 1$. Prove that
$$\lim_{n \longrightarrow \infty} \frac{x^n}{1 + x^n}$$
exists and can only possibly attain the values 0 or 1.

7. Let $K$ be a complete discretely valued field, say with non-trivial absolute value $|\cdot|_1$. Suppose $|\cdot|_2$ is some other non-trivial absolute value on $K$. Is $|\cdot|_1$ necessarily equivalent to $|\cdot|_2$? Give a proof or a counter-example.

8. Describe the higher unit filtration of $\mathcal{O}_{\mathbb{C}((t))}$.

9. Is every formal group law over $\mathbb{Z}_p$ isomorphic (over $\mathbb{Z}_p$) to a Lubin-Tate formal group law over $\mathbb{Z}_p$?

10. Describe all absolute values on $\mathbb{R}(t)$, up to equivalence of absolute values. Here $\mathbb{R}$ denotes the real numbers.

11. Is there a complete discretely valued field $K$ such that it has characteristic zero and its residue field is $\mathbb{F}_p^{sep}$. Give an example or prove its non-existence. Are all fields with these properties isomorphic? Give a counter-example or prove isomorphy.

12. Let $K$ be a complete discretely valued field $K$ such that its residue field is $\mathbb{F}_p^{sep}$. Is it always true that
$$\lim_{n \longrightarrow \infty} p^n = 0$$
in the topology of $K$? Can there be some other prime number $\ell$ such that $\lim_{n \longrightarrow \infty} \ell^n = 0$?

Please hand in your solutions to problems 1 and 2 in the lecture on Tuesday, 22nd of January. You may work in groups of at most three students. Problems 3-12 are easy questions to test your reflexes – do not hand them in. You should be able to answer them without much thinking. If you have trouble with some of them, look at them later.