

Problem Sheet 10

1. (4 points) Let A be a local Noetherian ring with perfect residue field k of prime characteristic p . Let F, F' be formal group laws such that the height of $F \otimes_A k$ is finite.

Prove that the natural map

$$\mathrm{Hom}_A(F, F') \rightarrow \mathrm{Hom}_k(F \otimes_A k, F' \otimes_A k)$$

is injective.

(Notation: $F \otimes_A k$ denotes the base change of the formal group law to k , i.e. you reduce all coefficients of the power series. Hint: It is enough to prove that if ϕ is a homomorphism that reduces to zero, then it's zero modulo $\mathfrak{m}_A^n$ for all n . (Why?) So we can replace A by $A/\mathfrak{m}_A^n$. Then $\mathfrak{m}_A^{n-1}$ is a finite dimensional k -vector space, so an element is zero if and only if it is killed by all functionals $\mathfrak{m}_A^{n-1} \rightarrow k$.)

2. (1+3 points) Let k be either a finite field of characteristic p or an algebraic closure $\bar{\mathbb{F}}_p$. Let $\mathcal{C}$ be the category of local Artinian rings with residue field k .

(a) Prove that $\mathcal{C}$ has all fibre products.

(b) Fix a formal group law F_0 over k . Consider the functor

$$\mathcal{F}: \mathcal{C} \rightarrow \mathcal{Sets}: A \mapsto \{F \text{ formal group law over } A: F \otimes_A k = F_0\} / \cong.$$

Prove that if $A \rightarrow C$ and $B \rightarrow C$ are morphisms in $\mathcal{C}$, with at least one of the two being surjective, then the natural map

$$\mathcal{F}(A \times_C B) \rightarrow \mathcal{F}(A) \times_{\mathcal{F}(C)} \mathcal{F}(B)$$

is bijective.

3. (4 points) Let H be a normal subgroup of a group G and let A be a G -module. Prove that the sequence below is exact

$$0 \rightarrow H^1(G/H, A^H) \xrightarrow{\mathrm{Inf}} H^1(G, A) \xrightarrow{\mathrm{Res}} H^1(H, A)$$

4. (4 points) (Continuation of exercise 3) Assume further that $H^i(H, A) = 0$ for $1 \leq i \leq q-1$. Prove that the sequence below is exact

$$0 \rightarrow H^q(G/H, A^H) \xrightarrow{\mathrm{Inf}} H^q(G, A) \xrightarrow{\mathrm{Res}} H^q(H, A).$$

(Hint: Induction on q , exercise 3 being the base case.) Both results could be obtained directly from the Hochschild–Serre spectral sequence, but your job is to give a direct proof.

Please hand in your solutions in the lecture on Tuesday, 15th of January. You may work in groups of at most three students.