Problem Sheet 1

1. (2+2 points) Let $x \in \mathbb{Q}_p$ with p-adic expansion $x = \sum_{n=N}^{\infty} a_n p^n$.
   (a) What is the p-adic expansion of $-x$?
   (b) Show that x lies in $\mathbb{Q}$ if and only if the sequence (a_n) is eventually periodic.

2. (1+1+1+1 points) *(Cardinalities)*
   (a) Prove that on a finite field every absolute value is trivial.
   (b) What is the cardinality of $\mathbb{Q}_p$?
   (c) Show that a complete valued field has never countable infinite cardinality, unless the absolute value is trivial. *(Hint: Use the Baire category theorem. If you don't know it, google it.)*
   (d) Define the *p-adic complex numbers*,
   $$\mathbb{C}_p := \widehat{\overline{\mathbb{Q}_p}},$$
   i.e. the completion of the algebraic closure of $\mathbb{Q}_p$. We will prove soon that this field is algebraically closed. Assuming this, deduce that there exists a field isomorphism $\mathbb{C}_p \simeq \mathbb{C}$, but no such isomorphism of topological fields.

3. (2+2 points) (a) Recall that $\hat{\mathbb{Z}}$ was defined to be $\varprojlim(\mathbb{Z}/n\mathbb{Z})$. Prove that
   $$\hat{\mathbb{Z}} \simeq \prod_{\ell \text{ prime}} \mathbb{Z}_\ell$$
   (b) Let $\mathbb{F}_q$ denote the finite field with q elements, and $\overline{\mathbb{F}_q}$ an algebraic closure. Under the identification $\mathrm{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right) \cong \hat{\mathbb{Z}}$, describe the fixed field of $\mathbb{Z}_\ell$ acting on $\overline{\mathbb{F}_q}$.

4. (4 points) Show that a valued field $(K, |\cdot|)$ is non-archimedean if and only if the set
   $$\{|n| \colon n \in \mathrm{im}(\mathbb{Z} \to K)\}$$
   is bounded. Deduce that on a field of positive characteristic, all absolute values are non-archimedean.

Please hand in you solutions in the lecture on Tuesday, 23rd of October. You may work in groups of at most three students.