THE TRANSCENDENTAL MOTIVE OF A SURFACE

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Contents

1. Introduction

1.1. The theory of pure motives was introduced by Grothendieck in the 1960s and since then it has become a powerful language to encode intersection-theoretic, cohomological, and arithmetic data of smooth, projective varieties. The theory is held together by a large web of conjectures, which are only proven for curves (Grothendieck standard conjectures, Kimura–O'Sullivan finite dimensionality, Bloch–Beilinson, see [17]). The higher dimensional cases are still open, thus the next step is to study the motive of a surface.

To this end, in [9], Kahn, Murre and Pedrini introduced a decomposition of the motive associated to a surface, building on ideas developed in Murre's paper [14]: Let S be a smooth, connected, projective surface over a field k and let $\mathfrak{h}(S)$ be its Chow motive, then

$$\mathfrak{h}(S) \cong \mathfrak{h}^0(S) \oplus \mathfrak{h}^1(S) \oplus \mathfrak{h}^2_{\mathrm{alg}}(S) \oplus \mathfrak{h}^2_{\mathrm{tr}}(S) \oplus \mathfrak{h}^3(S) \oplus \mathfrak{h}^4(S).$$

This decomposition is called a *refined Chow–Künneth decomposition* for S. In the present thesis, we study this decomposition, with emphasis on $\mathfrak{h}^2_{\mathrm{tr}}(S)$, the *transcendental motive* of S.

The transcendental motive $\mathfrak{h}^2_{\mathrm{tr}}(S)$ seems to be related to the Albanese kernel and to the transcendental cohomology of S. However, this is hard to make precise. Moreover, the crucial conjectures are not known for $\mathfrak{h}^2_{\mathrm{tr}}(S)$. A point to start the study of $\mathfrak{h}^2_{\mathrm{tr}}(S)$ is to analyze its endomorphisms $\mathrm{End}(\mathfrak{h}^2_{\mathrm{tr}}(S))$, or more generally $\mathrm{Hom}(\mathfrak{h}^2_{\mathrm{tr}}(S), \mathfrak{h}^2_{\mathrm{tr}}(S'))$ for surfaces S and S'.

The main goal of this thesis is to present and prove two descriptions of $\mathrm{Hom}(\mathfrak{h}^2_{\mathrm{tr}}(S), \mathfrak{h}^2_{\mathrm{tr}}(S'))$, following [9]:

$$\mathrm{Hom}(\mathfrak{h}^2_{\mathrm{tr}}(S), \mathfrak{h}^2_{\mathrm{tr}}(S')) \cong \frac{\mathrm{CH}^2(S \times S')_{\mathbb{Q}}}{J(S,S')}, \tag{1}$$

$$\mathrm{Hom}(\mathfrak{h}^2_{\mathrm{tr}}(S), \mathfrak{h}^2_{\mathrm{tr}}(S')) \cong \frac{\mathrm{T}\left(S'_{K(S)}\right)}{\mathrm{F}^1\,\mathrm{T}\left(S'_{K(S)}\right)}. \tag{2}$$

In (1), $J(S,S') \subseteq \mathrm{CH}^2(S \times S')_{\mathbb{Q}}$ is the subspace generated by cycles on $S \times S'$ that do not dominate both factors. By the construction of motives, $\mathrm{Hom}(\mathfrak{h}^2_{\mathrm{tr}}(S), \mathfrak{h}^2_{\mathrm{tr}}(S'))$ is a subspace of $\mathrm{CH}^2(S \times S')_{\mathbb{Q}}$. In a certain sense, description (1) is more natural, since the Chow group itself is defined as a quotient of the cycle group.

In (2), $\mathrm{T}(S'_K)$ is the Albanese kernel of $S'_{K(S)} = S' \times_k K(S)$ with rational coefficients. The subspace $\mathrm{F}^1\,\mathrm{T}(S'_{K(S)})$ is generated by the cycle classes of $S'_{K(S)}$ which are already defined on S'_L for some intermediate field $k \subseteq L \subseteq K(S)$ of transcendence degree at most 1. Description (2) witnesses the relationship of $\mathfrak{h}^2_{\mathrm{tr}}(S)$ with the Albanese kernel of S.

As an application, we consider curves on surfaces. The transcendental motive lets us characterize constant cycle curves, which were introduced by Huybrechts in [7]. Namely, a curve C on S is a constant cycle curve if and only if the morphism $\mathfrak{h}^1(C)(1) \to \mathfrak{h}^2_{\mathrm{tr}}(S) \oplus \mathfrak{h}^3(S)$ is zero. Moreover, we see that $\mathfrak{h}^2_{\mathrm{tr}}(S)$ is the only obstruction for $\mathrm{CH}_0(S)$ being a quotient of $\mathrm{CH}_0(C)$ for a curve C on S, which essentially reproves a classical theorem of Mumford, [12].

1.2. Organization of results. In Section 2, we define Chow–Künneth decompositions in general and introduce the refined Chow–Künneth decomposition and the transcendental motive of a surface. In Section 3 We analyze, how morphisms behave with respect to Chow–Künneth decompositions of curves and surfaces. In Section 4, this will be used to prove the main results about the endomorphisms of $\mathfrak{h}^2_{\mathrm{tr}}(S)$, and some easy consequences are drawn. In section 5 we describe the relationship to constant cycle curves and Mumford's theorem. We include two appendices: In the first we collect some facts about extension of scalars and Galois descent. In the second we give the foundations of Artin motives, which explains how to produce a motive out of a Galois representation

1.3. Notation and conventions. We work over an arbitrary field k. Its separable closure is denoted by k^s, its algebraic closure by $\bar{k}$. In this text, a *variety* is an integral scheme that is separated and of finite type over k. A *curve* is smooth, connected, projective variety of dimension 1, and a *surface* is a smooth, connected, projective variety of dimension 2. For a variety X, $\mathrm{CH}^k(X)$ denotes the Chow group of codimension k-cycles modulo rational equivalence, and $\mathrm{CH}^k(X)_0$ and $\mathrm{CH}^k(X)_{\mathrm{hom}}$ denote the subgroup of numerically respectively homologically trivial cycle classes. We fix a Weil cohomology theory H^* with coefficients in a field of characteristic 0, and write cl for the cycle class map $\mathrm{CH}^k(X) \to H^k(X)(k)$.

For an adequate equivalence relation $\sim$, we denote the corresponding category of pure motives by $\mathcal{M}_\sim$, as defined in [17]. A special role is taken by $\mathbb{1} = (\operatorname{Spec} k, \mathrm{id}, 0)$ and the Lefschetz motive $\mathbb{L} = (\operatorname{Spec} k, \mathrm{id}, -1)$. For a variety X, we write $\mathfrak{h}(X)$ for the motive $(X, \mathrm{id}, 0)$. For any motive M we write $M(-k)$ for $M \otimes \mathbb{L}^{\otimes k}$. For a Chow motive $M = (X, p, n)$, the Chow group $\mathrm{CH}^k(M)_\mathbb{Q}$ is by definition $\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathbb{L}^k, M) = p_* \mathrm{CH}^{k+n}(X)_\mathbb{Q}$. For a morphism $f\colon X \to Y$, we get correspondences $f_* = \Gamma_f\colon \mathfrak{h}(X) \to \mathfrak{h}(Y)$ and $f^* = \Gamma_f^t\colon \mathfrak{h}(Y)(d_X - d_Y) \to \mathfrak{h}(X)$.

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2. The refined Chow–Künneth Decomposition

The goal of this section is to prove the existence of the refined Chow–Künneth decomposition of a surface. In Sections 2.1 to 2.3 we briefly review Chow–Künneth decompositions in general, the case of curves and Murre's Picard and Albanese motive. This is enough to construct the refined Chow–Künneth decomposition in Section 2.4.

2.1. Suppose X and Y are varieties over k, and let $d = \dim(X)$. For a fixed Weil cohomology theory H^* we consider $\mathrm{Hom}_{gr}(H^*(X), H^*(Y))$, graded vector space morphisms between $H^*(X)$ and $H^*(Y)$. There is a natural isomorphism

$$\Phi\colon \mathrm{Hom}_{gr}(H^*(X), H^*(Y)) \xrightarrow{\simeq} H^{2d}(X \times Y)(d),$$

which is obtained by tensoring the Poincaré duality isomorphism $H^i(X)^* \cong H^{2d-i}(X)(d)$ with $H^i(Y)$ and then using the Künneth isomorphism. Note that the cohomology group on the right hand side contains all algebraic classes of degree d, i.e. cohomology classes of the form $\mathrm{cl}(Z)$ for $Z \in \mathrm{CH}^d(X \times Y)_\mathbb{Q}$. It is natural to ask for which morphisms $f \in \mathrm{Hom}_{gr}(H^*(X), H^*(Y))$ the cohomology class $\Phi(f)$ is algebraic.

We are interested in the case $Y = X$. For example we may take $f = \mathrm{id}$. One checks that $\Phi(\mathrm{id}) = \mathrm{cl}[\Delta_X]$, the fundamental class of the diagonal, which is algebraic by definition. On the other hand for $i = 0, \ldots, 2d$ let $f = \mathrm{pr}_{H^i(X)}$ be the projection onto the degree i part. We get cohomology classes $\pi^i := \Phi(\mathrm{pr}_{H^i(X)})$ satisfying $\sum \pi^i = \mathrm{cl}[\Delta_X]$, which are called the *Künneth components of the diagonal*. One of the famous standard conjectures states that they are algebraic, [17, 3.1.1], but this conjecture remains unsolved.

If the Künneth components are algebraic, say $\pi^i = \mathrm{cl}(p^i)$ for some correspondences $p^i \in \mathrm{CH}^d(X \times X)_\mathbb{Q}$, then we may ask wether the equality $\sum p^i = [\Delta_X]$ holds already on the level of rational Chow groups. Since the π^i are orthogonal projectors, we also ask if the p^i can be chosen to be orthogonal projectors, that is $p^i \circ p^i = p^i$ and $p^i \circ p^j = 0$ for $i \neq j$. This leads to the following definition.

Definition 2.1 ([15, Def 1.3.1])**.** A *Chow–Künneth decomposition for* X is a collection $p^0, \dots, p^{2d}$ of orthogonal projectors in $\mathrm{CH}^d(X \times X)_{\mathbb{Q}}$ such that $\mathrm{cl}(p^i) = \pi^i$ and $\sum p^i = [\Delta_X]$.

Murre proposed the following strengthening of the standard conjecture.

Conjecture 2.2 ([15, Conj. A])**.** *Every smooth, projective variety has a Chow–Künneth decomposition.*

For a given Chow–Künneth decomposition $p^0, \dots, p^{2d}$ we also get a decomposition of the motive $\mathfrak{h}(X)$, in every category of pure motives $\mathcal{M}_\sim$. Namely we set $\mathfrak{h}^i(X) = (X, p^i, 0)$ and get

$$\mathfrak{h}(X) \cong \bigoplus_{i=0}^{2d} \mathfrak{h}^i(X).$$

If the Weil cohomology theory factors through $\mathcal{M}_\sim$ (for example if $\mathcal{M}_\sim = \mathcal{M}_{\mathrm{rat}}$ is the category of Chow motives), then this gives a motivic splitting of the cohomology:

$$H^j(\mathfrak{h}^i(X)) = \begin{cases} H^i(X) & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

For geometrically connected X we have canonical isomorphisms $H^0(X) \cong H^0(\operatorname{Spec} k)$ and $H^{2d}(X) \cong H^{2d}(\mathbb{P}^d)$ (the first isomorphism is induced by the canonical map $X \to \operatorname{Spec} k$, the second follows by Poincaré duality). Thus we expect the motives $\mathfrak{h}^0(X)$ and $\mathfrak{h}^{2d}(X)$ to be rather easy. More specific, only if $x_0 \in X(k)$ we take $p^0 = [x_0 \times X], p^{2d} = [X \times x_0]$; if there is no k-rational point, then we might also pick any effective zero-cycle Z_0, renormalize it to have degree 1, and set $p^0 = \mathrm{pr}_1^*(Z_0), p^{2d} = \mathrm{pr}_2^*(Z_0)$. It is a classical computation, [17, 2.3(ii)], that $\mathfrak{h}^0(X) \coloneqq (X, p^0, 0) \cong \mathbb{1}$ and $\mathfrak{h}^{2d}(X) \coloneqq (X, p^{2d}, 0) \cong \mathbb{L}^d$, which corresponds to our expectations from cohomology. Indeed all Chow–Künneth decompositions which we use in the following will have p^0, p^{2d} as above.

2.2. (Curves). Before studying Chow–Künneth decompositions of surfaces in Section 2.4 we review the Chow–Künneth decompositions of curves. So suppose that C is a curve and define p^0, p^2 as above. We further set $p^1 = [\Delta_C] - p^0 - p^2$, and one easily checks that p^0, p^1, p^2 is a Chow–Künneth decomposition for C. Note that $(p^1)^t = p^1$.

The Chow groups of the Chow motives $\mathfrak{h}^i(C) \in \mathcal{M}_{\mathrm{rat}}$ are easy to compute. Since $\mathfrak{h}^0(C) \cong \mathbb{1}$ and $\mathfrak{h}^2(C) \cong \mathbb{L}$, their Chow groups are zero except for

$$\mathrm{CH}^0(\mathfrak{h}^0(C)) \cong \mathbb{Q}, \text{ generated by } [C] \in \mathrm{CH}^0(C)_{\mathbb{Q}},$$
$$\mathrm{CH}^1(\mathfrak{h}^2(C)) \cong \mathbb{Q}, \text{ generated by } [Z_0] \in \mathrm{CH}^1(C)_{\mathbb{Q}}.$$

Now, since the Chow groups of the $\mathfrak{h}^i(C)$ have to add up to the Chow groups of C, the only possibly nontrivial Chow group of $\mathfrak{h}^1(C)$ is $\mathrm{CH}^1(\mathfrak{h}^1(C)) \cong \mathrm{CH}^1(C)_{\mathbb{Q}}/[Z_0]$, which is isomorphic to $\mathrm{CH}^1(C)_{\mathbb{Q},0}$, the subspace generated by numerically trivial divisors.

For curves C, C', how does the space of morphisms $\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^i(C), \mathfrak{h}^j(C'))$ look like? Since $\mathfrak{h}^0(C) \cong \mathbb{1}$ and $\mathfrak{h}^2(C) \cong \mathbb{L}$, for $i \neq 1$ this question reduces to the computation of the Chow groups of $\mathfrak{h}^j(C')$. Note also that $\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^1(C), \mathfrak{h}^j(C')) = \mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^{2-j}(C'), \mathfrak{h}^1(C))^t$, since p^1 is symmetric. Hence if $j \neq 1$ we can again reduce to computing Chow groups. So the only exciting part is the isomorphism, [17, Thm. 2.7.2.(b)],

$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^1(C), \mathfrak{h}^1(C')) \cong \mathrm{Hom}_{\mathrm{AV}}(\mathrm{Jac}_C, \mathrm{Jac}_{C'}) \otimes \mathbb{Q},$$

where on the right side we have morphisms of abelian varieties between the Jacobians of C and C'. This link between Chow–Künneth components and abelian varieties will be further explored in the next section.

2.3. (Picard and Albanese motive). Following Scholl, [19], for a smooth, projective variety X of dimension $d \geq 2$ we construct projectors $p^1, p^{2d-1} \in \mathrm{CH}^d(X \times X)_{\mathbb{Q}}$ lifting the Künneth components π^1, π^{2d-1}, which we hope to complete to a full Chow–Künneth decomposition. (This will succeed for the case that X is a surface in Section 2.4.)

In the spirit of Lemma A.1 (ii), to construct the correspondences p^1, p^{2d-1} we may enlarge the ground field k by a finite Galois extension k'/k. After such an extension we may pick a smooth hyperplane section by Bertini's theorem, [5, Thm. II.8.18]. Let $h \in \mathrm{CH}^1(X)_{\mathbb{Q}}$ be its class and m its degree. We take an effective zero-cycle Z_0 with class the d-fold self-intersection $h^d \in \mathrm{CH}^d(X)$ to construct p^0 and p^{2d}, that is $p^0 = \frac{1}{m}\,\mathrm{pr}_1^*(Z_0)$ and $p^{2d} = (p^0)^t$.

Moreover note that the $(d-1)$-fold self-intersection $h^{d-1} \in \mathrm{CH}^{d-1}(X)$ is the class of a smooth connected curve C, see [5, Rem. III.7.9.1]. The inclusion $i\colon C \hookrightarrow X$ induces morphisms on the Picard and Albanese varieties, and we get a composition

$$\alpha\colon (\mathrm{Pic}^0_X)_{\mathrm{red}} \to (\mathrm{Pic}^0_C)_{\mathrm{red}} = \mathrm{Alb}_C \to \mathrm{Alb}_X\,.$$

It turns out that α is an isogeny, [17, Lem. 6.2.3 (1)]. Thus by the general theory of abelian varieties there is an isogeny $\beta\colon \mathrm{Alb}_X \to (\mathrm{Pic}^0_X)_{\mathrm{red}}$ and an integer $n \geq 1$ such that $\alpha \circ \beta$ and $\beta \circ \alpha$ are just multiplication by n. Now the universal properties of the Albanese and Picard varieties gives us a divisor class $\tilde{\beta} \in \mathrm{CH}^1(X \times X)$, namely $\tilde{\beta} = (\beta \circ \mathrm{AJ} \times \mathrm{id}_X)^* c_1(\mathcal{P})$, where $\mathrm{AJ}\colon X \to \mathrm{Alb}_X$ is the Abel–Jacobi map and $\mathcal{P}$ is the Poincaré bundle on $(\mathrm{Pic}^0_X)_{\mathrm{red}} \times X$.

We define an auxiliary correspondence

$$p^? := \frac{1}{n}\tilde{\beta} \cdot [C \times X]$$

and set

$$p^1 := p^? - \frac{1}{2}(p^?)^t \circ p^?, \qquad p^{2d-1} := (p^1)^t.$$

We call p^1 the *Picard projector* and p^{2d-1} the *Albanese projector* of X.[1] Set $\mathfrak{h}^i(X) = (X, p^i, 0)$ for $i = 0, 1, 2d-1, 2d$. Consequently, $\mathfrak{h}^1(X)$ is called the *Picard motive* and $\mathfrak{h}^{2d-1}(X)$ is called the *Albanese motive.*

Theorem 2.3 ([19, Thm. 4.4, 4.5] and [14])**.** *The correspondences $p^0, p^1, p^{2d-1}, p^{2d}$ are orthogonal idempotents lifting the Künneth components of the diagonal. Moreover*

(i) *The Chow groups of $\mathfrak{h}^1(X)$ and $\mathfrak{h}^{2d-1}(X)$ vanish, except*

$$\mathrm{CH}^1(\mathfrak{h}^1(X)) = \mathrm{CH}^1(X)_{\mathbb{Q},0} \cong (\mathrm{Pic}^0_X)_{\mathrm{red}}(k), \qquad \mathrm{CH}^d(\mathfrak{h}^{2d-1}(X)) = \mathrm{CH}^d(X)_{\mathbb{Q},0}/\ker(\mathrm{AJ}).$$

Here, $\mathrm{AJ}\colon \mathrm{CH}^d(X)_0 \to \mathrm{Alb}_X$ *is the Abel–Jacobi map.*

(ii) (Hard Lefschetz) *The following composition is an isomorphism:*

$$\mathfrak{h}^1(X) \hookrightarrow \mathfrak{h}(X) \xrightarrow{i_* \circ i^*} \mathfrak{h}(X)(d-1) \twoheadrightarrow \mathfrak{h}^{2d-1}(X)(d-1)$$

(iii) *For two varieties X, Y of dimension d_X, d_Y with Picard and Albanese projectors as above we have*

$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^1(X), \mathfrak{h}^1(Y)) \cong \mathrm{Hom}_{\mathrm{AV}}((\mathrm{Pic}^0_X)_{\mathrm{red}}, (\mathrm{Pic}^0_Y)_{\mathrm{red}}) \otimes \mathbb{Q},$$
$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^{2d_X-1}(X)(d_Y - d_X), \mathfrak{h}^{2d_Y-1}(Y)) \cong \mathrm{Hom}_{\mathrm{AV}}(\mathrm{Alb}_X, \mathrm{Alb}_Y) \otimes \mathbb{Q}\,.$$

Note that in (i) the Chow groups can be described in terms of abelian varieties: On one hand $\mathrm{CH}^1(X)_{\mathbb{Q},0} \cong (\mathrm{Pic}^0_X)_{\mathrm{red}}(k) \otimes \mathbb{Q}$. On the other hand, $\mathrm{CH}^d(X)_{\mathbb{Q},0}/\ker(\mathrm{AJ}) \cong \mathrm{Alb}_X(k) \otimes \mathbb{Q}$, since the Abel-Jacobi map is always surjective, see [20, Lem. 12.11] for a proof over $\mathbb{C}$.

In (iii), the isomorphism does not only hold for morphisms in the category $\mathcal{M}_{\mathrm{rat}}$ of Chow motives, but rather in $\mathcal{M}_\sim$ for any adequate equivalence relation $\sim$. This will be proved along the way in Remark 3.9. We further remark that one does not loose any information on the left hand side when tensoring with $\mathbb{Q}$, as the following lemma shows:

Lemma 2.4. *For two abelian varieties A, B the group $\mathrm{Hom}_{\mathrm{AV}}(A, B)$ is torsion-free.*

Proof. If $\phi\colon A \to B$ satisfies $n\phi = 0$, then the image of ϕ is connected and lies in the finite set of n-torsion elements in B, [13, II.6. App. 2]. This shows $\phi = 0$. □

Remark 2.5. The correspondence p^1 can be taken to be a cycle with support in $C \times X$, and p^3 can be taken to be a cycle with support in $X \times C$. Indeed, $p^? = \frac{1}{n}\tilde{\beta} \cdot [C \times X]$, and by the moving lemma we may assume that $\tilde{\beta}$ intersects $C \times X$ properly, so that $p^?$ is defined as a cycle with support in $C \times X$. Similarly we replace $(p^?)^t$ by a rationally equivalent cycle, so that $(p^?)^t \circ p^?$ is a cycle with support in $C \times X$. The results for $p^?$ and $(p^?)^t \circ p^?$ together imply the result for $p^1 = p^? - \frac{1}{2}(p^?)^t \circ p^?$. Taking transposes yields the result for p^{2d-1}.

[1] In [19], Scholl actually defines $p^1 := p^? - \frac{1}{2}p^? \circ (p^?)^t$. With our definition, all the proofs still go through, but the geometry of p^1 is easier. Namely, Remark 2.5 holds.

2.4. (Surfaces). The results above apply to a surface S, so we get orthogonal projectors p^0, p^1, p^3, p^4. Now we just set

$$p^2 = [\Delta_S] - p^0 - p^1 - p^3 - p^4,$$

and $p^0, \dots, p^4$ is a full Chow–Künneth decomposition of S.

The Chow groups fit into the following table

	$\mathfrak{h}^0(S)$	$\mathfrak{h}^1(S)$	$\mathfrak{h}^2(S)$	$\mathfrak{h}^3(S)$	$\mathfrak{h}^4(S)$
CH^0	$\mathbb{Q}$				
CH^1		$(\mathrm{Pic}^0_S)_{\mathrm{red}}(k) \otimes \mathbb{Q}$	$\mathrm{NS}(S)_{\mathbb{Q}}$		
CH^2			$\ker(\mathrm{AJ}) \otimes \mathbb{Q}$	$\mathrm{Alb}_S(k) \otimes \mathbb{Q}$	$\mathbb{Q}$.

We get this from Theorem 2.3 and considerations similar to the case of curves. Here, $\mathrm{NS}(S)$ is the Néron–Severi group of S and the empty spots are zero.

Interestingly, $\mathfrak{h}^2(S)$ is the only motive encoding two nontrivial pieces of information. This is reflected by the fact that p^2 splits into an algebraic and a transcendental part as follows.

Theorem 2.6 ([9, Pro. 2.3])**.** *There exists a decomposition*

$$p^2 = p^2_{\mathrm{alg}} + p^2_{\mathrm{tr}} \in \mathrm{CH}^2(S \times S)_{\mathbb{Q}}$$

into projectors that are mutually orthogonal and orthogonal to all $p^j, j \neq 2$, such that the motive $(S, p^2_{\mathrm{alg}}, 0)$ is isomorphic to $\underline{\mathrm{NS}}(S_{k^s})_{\mathbb{Q}} \otimes \mathbb{L}$. Here, $\underline{\mathrm{NS}}(S_{k^s})_{\mathbb{Q}}$ is the Artin motive associated to the geometrical, rational Néron–Severi group $\mathrm{NS}(S_{k^s})_{\mathbb{Q}}$.

Definition 2.7. We call $p^0, p^1, p^2_{\mathrm{alg}}, p^2_{\mathrm{tr}}, p^3, p^4$ a *refined Chow–Künneth decomposition of* S. We set $\mathfrak{h}^2_{\mathrm{alg}}(S) = (S, p^2_{\mathrm{alg}}, 0)$ and $\mathfrak{h}^2_{\mathrm{tr}}(S) = (S, p^2_{\mathrm{tr}}, 0)$, so that we get a splitting

$$\mathfrak{h}^2(S) = \mathfrak{h}^2_{\mathrm{alg}}(S) \oplus \mathfrak{h}^2_{\mathrm{tr}}(S).$$

We call $\mathfrak{h}^2_{\mathrm{tr}}(S)$ the *transcendental motive of* S.

By the construction which will be given in the proof we get $(p^2_{\mathrm{alg}})^t = p^2_{\mathrm{alg}}$, so that also $(p^2_{\mathrm{tr}})^t = p^2_{\mathrm{tr}}$. Moreover, the splitting $p^2 = p^2_{\mathrm{alg}} + p^2_{\mathrm{tr}}$ is unique. We will prove this in Corollary 2.10.

Remark 2.8. The Chow groups split as follows: By construction (see Proposition B.3)

$$\mathrm{CH}^1(\mathfrak{h}^2_{\mathrm{alg}}(S)) = \mathrm{NS}(S)_{\mathbb{Q}}, \;\; \mathrm{CH}^2(\mathfrak{h}^2_{\mathrm{tr}}(S)) = \ker(\mathrm{AJ}) \otimes \mathbb{Q},$$

and all other Chow groups vanish.

The cohomology also splits nicely: *A fortiori*, the only nontrivial cohomology group is H^2. We have that $H^2(\mathfrak{h}^2_{\mathrm{alg}}(S)) = H^2_{\mathrm{alg}}(S)$, the cohomology generated by the algebraic classes (up to a Tate twist). We define the *transcendental cohomology* $H^2_{\mathrm{tr}}(S)$ to be $H^2(\mathfrak{h}^2_{\mathrm{tr}}(S))$. It is the orthogonal complement to $H^2_{\mathrm{alg}}(S)$ inside $H^2(S)$, with respect to the cup-product pairing.

Proof of Theorem 2.6. For clarity we proceed in two steps:

(i) We prove the theorem under the additional assumptions that the Galois action on $\mathrm{NS}(S_{k^s})_{\mathbb{Q}}$ is trivial and $\rho(S_{k^s}) = \rho(S)$. (This is enough for the reader who is only interested in the case of a separatedly closed ground field k.) Choose an orthogonal basis $D_1, \ldots, D_\rho$ of the Néron–Severi group $\mathrm{NS}(S)_{\mathbb{Q}}$ with respect to the intersection product. Consider the map $s\colon \mathrm{NS}(S)_{\mathbb{Q}} \to \mathrm{CH}^1(S)_{\mathbb{Q}}$ induced by taking the first Chow group of the inclusion $\mathfrak{h}^2(S) \hookrightarrow \mathfrak{h}(S)$. It is a section to the natural projection $\mathrm{CH}^1(S)_{\mathbb{Q}} \to \mathrm{NS}(S)_{\mathbb{Q}}$. This provides lifts $\ell_i \coloneqq s(D_i) \in \mathrm{CH}^1(S)_{\mathbb{Q}}$.

We define the correspondences $\alpha_i = \frac{1}{(\ell_i)^2}\ell_i \times \ell_i \in \mathrm{CH}^1(S \times S)_{\mathbb{Q}}$; they are orthogonal projectors. Moreover we have

$$p^2 \circ \alpha_i = \alpha_i \circ p^2 = \alpha_i. \tag{1}$$

Indeed by the definition of s we have $p^2_*(\ell_i) = \ell_i$ and by Lieberman's lemma, [17, Lem. 2.1.3.], we obtain for instance $p^2 \circ \alpha_i = (\mathrm{id} \times p^2)_*(\alpha_i) = \frac{1}{(\ell_i)^2}\left[\ell_i \times p^2_*\ell_i\right] = \alpha_i$. Also

$$p^j \circ \alpha_i = \alpha_i \circ p^j = 0 \text{ for } j \neq 2; \tag{2}$$

this follows from equation (1), since $p^j \circ p^2 = p^2 \circ p^j = 0$.

We get the correspondence $p^2_{\mathrm{alg}} = \sum_i \alpha_i$. Since the α_i are orthogonal projectors, p^2_{alg} is also a projector. By equation (2), p^2_{alg} is orthogonal to all p^j for $j \neq 2$. Moreover equation (1) implies that $p^2_{\mathrm{tr}} \coloneqq p^2 - p^2_{\mathrm{alg}}$ is an orthogonal projector, the claimed properties about it follow at once.

By assumption the Artin motive $\underline{\mathrm{NS}}(S_{k^s})_{\mathbb{Q}}$ is isomorphic to $\mathbb{1}^{\oplus\rho}$. So to prove that $(S, p^2_{\mathrm{alg}}, 0) \cong \underline{\mathrm{NS}}(S_{k^s})_{\mathbb{Q}} \otimes \mathbb{L}$ it is enough to show that $(S, \alpha_i, 0) \cong \mathbb{L}$. Indeed, we can view the ℓ_i as correspondences $f_i = [\ell_i \times \mathrm{pt}] \in \mathrm{CH}^1(X \times \mathrm{pt})_{\mathbb{Q}}$. (We write "pt" for the scheme $\operatorname{Spec} k$ to save space.) The inverse is given by $g_i = \frac{1}{(\ell_i)^2} f_i^t$. We calculate

$$g_i \circ f_i = \frac{1}{(\ell_i)^2}\,\mathrm{pr}_{13*}\left([\ell_i \times \mathrm{pt} \times S] \cdot [S \times \mathrm{pt} \times \ell_i]\right) = \frac{1}{(\ell_i)^2}[\ell_i \times S] \cdot [S \times \ell_i] = \alpha_i,$$

and similar $f_i \circ g_i = \mathrm{id}_{\mathbb{L}}$. This proves that f_i, g_i are mutually inverse isomorphisms $(S, \alpha_i, 0) \cong \mathbb{L}$.

(ii) For general k we use a descent argument: By Proposition B.1, there always is a finite Galois extension k'/k such that the $\mathrm{Gal}(k^s/k)$-action on $\mathrm{NS}(S_{k^s})_{\mathbb{Q}}$ factors over $\mathrm{Gal}(k'/k)$. Thus the action of $\mathrm{Gal}(k^s/k')$ is trivial and, enlarging k' if necessary, we might assume that $\rho(S_{k^s}) = \rho(S_{k'})$. By (i) the orthogonal projectors $p^2_{\mathrm{alg},S_{k'}}$ and $p^2_{\mathrm{tr},S_{k'}}$ exist, and by Proposition A.1 they descend to orthogonal projectors $p^2_{\mathrm{alg},S}$ and $p^2_{\mathrm{tr},S}$ on S satisfying $p^2_{\mathrm{alg},S} + p^2_{\mathrm{tr},S} = p^2_S$.

Observe further that $p^2_{\mathrm{alg},S_{k'}}$ is already $\mathrm{Gal}(k'/k)$-invariant, because the ℓ_i form an orthogonal basis with respect to the $\mathrm{Gal}(k'/k)$-invariant intersection pairing. Thus actually $p^2_{\mathrm{alg},S_{k'}} = \pi^*(p^2_{\mathrm{alg},S})$ and the injectivity of π^* implies that $(S, p^2_{\mathrm{alg},S}, 0) \cong \underline{\mathrm{NS}}(S_{k^s})_{\mathbb{Q}} \otimes \mathbb{L}$. ☐

Finally, let us collect what can be immediately said about morphisms between transcendental and algebraic motives of surfaces. The study of $\mathrm{Hom}(\mathfrak{h}^2_{\mathrm{tr}}(S), \mathfrak{h}^2_{\mathrm{tr}}(S'))$ is much harder and is postponed to Section 4.

Proposition 2.9. *Let S and S' be surfaces with fixed refined Chow–Künneth decompositions.*
(i) $\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^2_{\mathrm{alg}}(S), \mathfrak{h}^2_{\mathrm{tr}}(S')) = \mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^2_{\mathrm{tr}}(S), \mathfrak{h}^2_{\mathrm{alg}}(S')) = 0,$
(ii) $\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^2_{\mathrm{alg}}(S), \mathfrak{h}^2_{\mathrm{alg}}(S')) \cong \mathrm{Hom}_{\mathrm{RepG}(k^s/k)}(\mathrm{NS}(S_{k^s})_{\mathbb{Q}}, \mathrm{NS}(S'_{k^s})_{\mathbb{Q}}).$

Proof. (i) Note that the refined Chow-Künneth decomposition is stable under extension of scalars. We can choose a field extension such that $\mathrm{NS}(S_{k^s})_{\mathbb{Q}}$ is a trivial $\mathrm{Gal}(k^s/k')$-representation (for example $k' = k^s$). Then $\mathfrak{h}^2_{\mathrm{alg}}(S_{k'}) \cong \mathbb{L}^{\oplus\rho}$. From $\mathrm{Hom}(\mathbb{L}, \mathfrak{h}^2_{\mathrm{tr}}(S'_{k'})) = \mathrm{CH}^1(\mathfrak{h}^2_{\mathrm{tr}}(S'_{k'}))_{\mathbb{Q}} = 0$, it follows that $\mathrm{Hom}(\mathfrak{h}^2_{\mathrm{alg}}(S_{k'}), \mathfrak{h}^2_{\mathrm{tr}}(S'_{k'})) = 0$. By the injectivity of $\mathrm{Hom}(\mathfrak{h}^2_{\mathrm{alg}}(S), \mathfrak{h}^2_{\mathrm{tr}}(S')) \to \mathrm{Hom}(\mathfrak{h}^2_{\mathrm{alg}}(S_{k'}), \mathfrak{h}^2_{\mathrm{tr}}(S'_{k'}))$ the first group is also zero. Finally, we get $\mathrm{Hom}(\mathfrak{h}^2_{\mathrm{tr}}(S), \mathfrak{h}^2_{\mathrm{alg}}(S')) = 0$ by dualizing and interchanging the roles of S and S'.

(ii) Use $\mathrm{Hom}(\mathfrak{h}^2_{\mathrm{alg}}(S), \mathfrak{h}^2_{\mathrm{alg}}(S')) \cong \mathrm{Hom}(\underline{\mathrm{NS}}(S_{k^s})_{\mathbb{Q}}, \underline{\mathrm{NS}}(S'_{k^s})_{\mathbb{Q}})$ and Proposition B.2. □

Corollary 2.10. *The decomposition $p^2 = p^2_{\mathrm{alg}} + p^2_{\mathrm{tr}}$ of Theorem 2.6 is unique.*

Proof. By part (i) of Proposition 2.9 we have $\mathrm{End}(\mathfrak{h}^2(S)) \cong \mathrm{End}(\mathfrak{h}^2_{\mathrm{alg}}(S)) \times \mathrm{End}(\mathfrak{h}^2_{\mathrm{tr}}(S))$. By part (ii), $\mathrm{End}(\mathfrak{h}^2_{\mathrm{alg}}(S)) \cong \mathrm{End}(\mathrm{NS}(S_{k^s})_{\mathbb{Q}})$, so $\mathrm{End}(\mathfrak{h}^2_{\mathrm{tr}}(S))$ can be intrinsically characterized as the subring of $\mathrm{End}(\mathfrak{h}^2(S))$ consisting of the endomorphisms that induce the zero map on $\mathrm{NS}(S_{k^s})_{\mathbb{Q}}$. So p^2_{tr} is uniquely determined by being the projector corresponding to the identity of $\mathrm{End}(\mathfrak{h}^2_{\mathrm{tr}}(S))$, and then of course $p^2_{\mathrm{alg}} = p^2 - p^2_{\mathrm{tr}}$. □

3. Morphisms between Chow–Künneth components

In this section, we discuss how morphisms of motives respect Chow–Künneth decompositions. The main result is Proposition 3.5, which is conditional to some conjectures of Murre. The conjectures are discussed in Section 3.2. To apply them to motives we have to introduce a Chow–Künneth decomposition on the product of two varieties in Section 3.1. In Section 3.3, we study how this applies to curves and surfaces.

3.1. Suppose in the following that X and Y are connected, smooth, projective varieties of dimension d_X and d_Y with Chow–Künneth decompositions. The space of morphisms between the motives $(X, p, 0)$ and $(Y, q, 0)$ is a subspace of $\mathrm{CH}^{d_X}(X \times Y)_{\mathbb{Q}}$. Thus, to analyze it, it is helpful that $X \times Y$ itself inherits a Chow–Künneth decomposition from X and Y, as follows:

Definition 3.1. Suppose that $p^0_X, \ldots, p^{2d_X}_X$ and $p^0_Y, \ldots, p^{2d_Y}_Y$ are Chow–Künneth decompositions for X and Y. For $X \times Y$ we define the *product Chow–Künneth decomposition* by

$$p^k_{X\times Y} = \sum_{i+j=k} \tilde{p}^i_X \times p^j_Y,$$

with $\tilde{p}^i = \left(p^{2d_X - i}\right)^t$. Here, we tacitly identify $X \times X \times Y \times Y$ with $X \times Y \times X \times Y$ by interchanging the second and third factors.

Remark 3.2. Since the Künneth components of the diagonal satisfy $\pi^i = \left(\pi^{2d-i}\right)^t$, the $\tilde{p}^i$ also give a Chow–Künneth decomposition for X. However, $p^i = \left(p^{2d-i}\right)^t$ in most cases of interest to us, for example in the case of surfaces as discussed in Section 2.4. Thus, the reader may feel free to ignore the difference between p^i and $\tilde{p}^i$.

The $p^k_{X\times Y}$ indeed form a Chow–Künneth decomposition for $X \times Y$: The formula $(\alpha \times \beta) \circ (\alpha' \times \beta') = (\alpha \circ \alpha') \times (\beta \circ \beta')$ shows that the $\tilde{p}^i_X \times p^j_Y$ are mutually orthogonal projectors, hence the $p^k_{X\times Y}$ are also mutually orthogonal projectors. Moreover, the Künneth formula shows that the $p^k_{X\times Y}$ lift the Künneth components of the diagonal.

3.2. To get a better grip on the Chow–Künneth decomposition, Murre formulated the following conjectures.

Conjecture 3.3 ([15, Conj. B, D])**.** *Suppose that X has a Chow–Künneth decomposition $p^0, \dots, p^{2d}$. Let $k = 0, \dots, d$.*

(A) *The projectors $p^0, \dots, p^{k-1}$ and $p^{2k+1}, \dots, p^{2d}$ act as zero on $\mathrm{CH}^k(X)_{\mathbb{Q}}$.*

(B) *We have*

$$\ker\left(p^{2k}_*\colon\ \mathrm{CH}^k(X)_{\mathbb{Q}} \to \mathrm{CH}^k(X)_{\mathbb{Q}}\right) = \mathrm{CH}^k(X)_{\mathrm{hom},\mathbb{Q}}.$$

Remark 3.4. We collect some cases where the conjectures are known.

(i) In (B), one always has $\ker\left(p^{2k}_*\right) \subseteq \mathrm{CH}^k(X)_{\mathbb{Q},\mathrm{hom}}$: For $Z \in \ker\left(p^{2k}_*\right)$ we have

$$\mathrm{cl}(Z) = \pi^{2k}_*(\mathrm{cl}(Z)) = \mathrm{cl}(p^{2k}_*(Z)) = 0,$$

since π^{2k}_* is the identity on $H^{2k}(X)$.

(ii) In the case of curves both conjectures are easily checked, [15, Sect. 2.2.].

(iii) If $p^0, p^1, p^{2d-1}, p^{2d}$ are constructed as in Section 2.3, then the conjectures hold for $\mathrm{CH}^1(X)_{\mathbb{Q}}$: For (A) we have to show that $p^i_*(D) = 0$ for every divisor D and every $i \neq 1, 2$. Since π^i acts as zero on $H^2(X)$, the divisor $p^i_*(D)$ is homologically trivial, which for divisors is equivalent to being numerically trivial, [11]. By Theorem 2.3 (i), it lies in the image of the projector p^1_*, so $p^i_*(D) = p^1_*(p^i_*(D)) = (p^1 \circ p^i)_*(D) = 0$, using that the p^i are orthogonal. For (B) we take a homologically trivial divisor D, and note that $p^2_*(D)$ is also homologically trivial. As before we conclude that $p^2_*(D) = p^1_*(p^2_*(D)) = 0$. See also [15, Lemma 2.1.1].

(iv) The result above applies to Chow–Künneth decompositions for surfaces as described in Section 2.4. In this case the conjectures also hold for $\mathrm{CH}^2(X)_{\mathbb{Q}}$: (A) follows from $\mathrm{CH}^2(\mathfrak{h}^0(S))_{\mathbb{Q}} = \mathrm{CH}^2(\mathfrak{h}^1(S))_{\mathbb{Q}} = 0$, and (B) follows from $\mathrm{CH}^2(S)_{0,\mathbb{Q}} = \mathrm{CH}^2(S)_{\mathrm{hom},\mathbb{Q}}$.

(v) Murre checked that the conjectures hold for products of the form $X = C \times S$ of a curve C and a surface S, equipped with the product Chow–Künneth decomposition, [16].

(vi) Kimura proved that (A) is true for the product of two surfaces with the product Chow–Künneth decomposition, [10]. In the same paper, (B) is proven for certain products of the form $X = C_1 \times C_2 \times S$ under further Hodge theoretic assumptions. In general, (B) is not known for the product of surfaces.

3.3. Assuming Conjecture 3.3, the group $\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^i(X), \mathfrak{h}^j(Y))$ is rather simple for $i \leq j$.

Proposition 3.5 ([8, Prop. 5.8])**.** *Suppose that X, Y have Chow–Künneth decompositions and $X \times Y$ is equipped with the product Chow–Künneth decomposition (see Definition 3.1).*

(i) *If $X \times Y$ satisfies* (A) *in Conjecture 3.3 for* $\mathrm{CH}^{d_X}(X \times Y)_{\mathbb{Q}}$*, then for all $i < j$*

$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^i(X), \mathfrak{h}^j(Y)) = 0.$$

(ii) *If $X \times Y$ satisfies* (B) *in Conjecture 3.3 for* $\mathrm{CH}^{d_X}(X \times Y)_{\mathbb{Q}}$*, then for all i*

$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^i(X), \mathfrak{h}^i(Y)) = \mathrm{Hom}_{\mathcal{M}_{\mathrm{hom}}}(\mathfrak{h}^i(X), \mathfrak{h}^i(Y)).$$

Remark 3.6. We may rewrite the assertions of the theorem in formulas as

$$\text{(i')} \qquad p_Y^j \circ f \circ p_X^i = 0 \ \text{ for } f \in \mathrm{CH}^{d_X}(X \times Y)_{\mathbb{Q}},$$

$$\text{(ii')} \qquad p_Y^i \circ f \circ p_X^i = 0 \ \text{ for } f \in \mathrm{CH}^{d_X}(X \times Y)_{\mathbb{Q},\mathrm{hom}}.$$

Proof. Take some $f \in \mathrm{CH}^{d_X}(X \times Y)_{\mathbb{Q}}$. By Lieberman's Lemma, [17, Lem. 2.1.3.], we have

$$p_Y^j \circ f \circ p_X^i = ((p_X^i)^t \times p_Y^j)_*(f) = (\tilde{p}_X^{2d_X - i} \times p_Y^j)_*(f).$$

We set $k = 2d_X - i + j$. If $i < j$, then $k > 2d_X$, so $(p^k_{X\times Y})_*(f) = 0$ by (A) in Conjecture 3.3. Now $p^k_{X\times Y}$ is the sum of mutually orthogonal projectors, one of which is $\tilde{p}_X^{2d_X-i} \times p_Y^j$. Thus also $(\tilde{p}_X^{2d_X-i} \times p_Y^j)_*(f) = 0$, which proves (i'). Similarly, if f is homologically trivial and $i = j$, then $k = 2d_X$ and so by (B) in Conjecture 3.3 we get $(\tilde{p}_X^{2d_X-i} \times p_Y^j)_*(f) = 0$, proving (ii'). □

Recall from Remark 3.4 (v) that Murre has proven Conjecture 3.3 for the product of a curve and a surface in [16]. Thus we get the following result:

Corollary 3.7. *For a curve C and a surface S we have*

$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^i(C), \mathfrak{h}^j(S)) = \begin{cases} 0 & \text{for } i < j \\ \mathrm{Hom}_{\mathcal{M}_{\mathrm{hom}}}(\mathfrak{h}^i(C), \mathfrak{h}^i(S)) & \text{for } i = j. \end{cases}$$

For the product of two surfaces Conjecture 3.3 (B) is not known, so Proposition 3.5 (ii) does not apply. However, it is interesting to see that it holds unconditionally for the Picard and Albanese motives:

Proposition 3.8. *Suppose that X, Y have Chow–Künneth decomposition with the Picard and Albanese projectors. Then*

$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^1(X), \mathfrak{h}^1(Y)) = \mathrm{Hom}_{\mathcal{M}_{\mathrm{hom}}}(\mathfrak{h}^1(X), \mathfrak{h}^1(Y)),$$
$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^{2d_X-1}(X)(d_X-d_Y), \mathfrak{h}^{2d_Y-1}(Y)) = \mathrm{Hom}_{\mathcal{M}_{\mathrm{hom}}}(\mathfrak{h}^{2d_X-1}(X)(d_X-d_Y), \mathfrak{h}^{2d_Y-1}(Y)).$$

Proof. By Theorem 2.3 (ii) we have an isomorphism $i_* \circ i^* \colon \mathfrak{h}^1(X) \cong \mathfrak{h}^{2d_X-1}(d_X-1)$ preserving homological equivalence. So we get

$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^1(X), \mathfrak{h}^1(Y)) \cong \mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^{2d_X-1}(X)(d_X-1), \mathfrak{h}^1(Y)) = p_Y^1 \circ \mathrm{CH}^1(X \times Y)_{\mathbb{Q}} \circ p_X^{2d_X-1}.$$

Take $f \in p_Y^1 \circ \mathrm{CH}^1(X \times Y)_{\mathbb{Q}} \circ p_X^{2d_X-1}$. Use Lieberman's lemma, [17, Lem. 2.1.3.], and that $p_X^1 \times p_Y^1$ is an orthogonal summand of $p_{X\times Y}^2$ from the product Chow–Künneth decomposition, to get

$$f = (p_X^1 \times p_Y^1)_*(f) = (p_{X\times Y}^2)_*(f).$$

Note that by Remark 3.4 (iii), part (B) of Conjecture 3.3 holds true, so that if f is homologically trivial, then $f = (p_{X\times Y}^2)_*(f) = 0$. This shows the first assertion, the seconds follow by dualizing. □

Remark 3.9. This implies that for every adequate equivalence relation $\sim$ the morphisms $\mathrm{Hom}_{\mathcal{M}_\sim}(\mathfrak{h}^1(X), \mathfrak{h}^1(Y))$ and $\mathrm{Hom}_{\mathcal{M}_\sim}(\mathfrak{h}^{2d_X-1}(X)(d_X-d_Y), \mathfrak{h}^{2d_Y-1}(Y))$ are the same. Indeed, we can view $\mathrm{Hom}_{\mathcal{M}_\sim}(\mathfrak{h}^1(X), \mathfrak{h}^1(Y))$ as a certain subspace of $\mathrm{CH}^1(X\times Y)_{\mathbb{Q}}/\sim$, as in the proof of the proposition. Since on $\mathbb{Q}$-divisors homological and numerical equivalence coincide, [11], we have shown that $\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^1(X), \mathfrak{h}^1(Y)) = \mathrm{Hom}_{\mathcal{M}_{\mathrm{num}}}(\mathfrak{h}^1(X), \mathfrak{h}^1(Y))$. We conclude by noting that rational equivalence is the finest and numerical equivalence is the coarsest adequate equivalence relation. By Theorem 2.3 we can always describe these morphism spaces in terms of maps of abelian varieties.

Propositions 3.5 and 3.8 together with Remark 3.4 (v) imply the following result.

Corollary 3.10. *For surfaces S and S' we have*

$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^i(S), \mathfrak{h}^j(S')) = \begin{cases} 0 & \text{for } i<j \\ \mathrm{Hom}_{\mathcal{M}_{\mathrm{hom}}}(\mathfrak{h}^i(S), \mathfrak{h}^i(S')) & \text{for } i=j\neq 2. \end{cases}$$

The following natural question remains open. We will come back to it in Corollary 4.7.

Question 3.11. *Is* $\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^2(S), \mathfrak{h}^2(S')) = \mathrm{Hom}_{\mathcal{M}_{\mathrm{hom}}}(\mathfrak{h}^2(S), \mathfrak{h}^2(S'))$*?*

4. Description of $\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^2_{\mathrm{tr}}(S), \mathfrak{h}^2_{\mathrm{tr}}(S'))$

We give two descriptions of $\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^2_{\mathrm{tr}}(S), \mathfrak{h}^2_{\mathrm{tr}}(S'))$ for surfaces S and S', see Theorems 4.1 and 4.6. Sections 4.1 and 4.3 are devoted to their proofs. Section 4.2 introduces a filtration on $\mathrm{CH}^2(S'_{K(S)})_{\mathbb{Q}}$, which is needed for the second description. We collect some corollaries of the descriptions in Section 4.4.

4.1. We start with a lemma about the action of the transcendental projector p^2_{tr} on zero-cycles.

Lemma 4.1. *Let C be a smooth hyperplane section of S. For every zero-cycle Z on S there is a zero-cycle W with support in C such that as elements of* $\mathrm{CH}^2(S)_{\mathbb{Q}}$

$$(p^2_{\mathrm{tr}})_*(Z) = Z + W.$$

Proof. We have the decomposition

$$[\Delta_S] = p^0 + p^1 + p^2_{\mathrm{alg}} + p^2_{\mathrm{tr}} + p^3 + p^4.$$

The projectors $p^0, p^1, p^2_{\mathrm{alg}}$ act as 0 on zero-cycles, hence

$$(*) \qquad Z = (p^2_{\mathrm{tr}})_*(Z) + p^3_*(Z) + p^4_*(Z).$$

By Remark 2.5 we may take p^3 to be a cycle with support contained in $S \times C$. Using the moving lemma, we replace Z by a cycle such that $W' := p^3_*(Z) = (\mathrm{pr}_2)_*([Z \times S] \cdot p^3)$ is defined as a cycle, which then *a fortiori* has support in C.

In the same spirit, $p^4 = [S \times Z_0]$, where Z_0 lies in the class of the self intersection $[C] \cdot [C]$ (see Section 2.3). By the moving lemma we thus assume that Z_0 has support in C, so that p^4 has support in $S \times C$. Referring to the moving lemma once more, we replace Z by a cycle such that $W'' := p^4_*(Z)$ is defined as a cycle with support contained in C.

Now we set $W = -(W' + W'')$ and plug this into equation $(*)$ to obtain the result. □

For two surfaces S and S', consider the map

$$\Phi\colon\ \mathrm{CH}^2(S \times S')_{\mathbb{Q}} \to \mathrm{CH}^2(S \times S')_{\mathbb{Q}} \colon f \mapsto p^2_{\mathrm{tr},S'} \circ f \circ p^2_{\mathrm{tr},S}.$$

By definition $\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^2_{\mathrm{tr}}(S), \mathfrak{h}^2_{\mathrm{tr}}(S'))$ is the image of Φ. To understand it we need to find generators for the kernel.

Definition 4.2. We write $J(S, S')$ for the subspace of $\mathrm{CH}^2(S, S')_{\mathbb{Q}}$ generated by those cycles that do not dominate both S and S'.

See Appendix A for a brief discussion of what is meant by a cycle dominating a factor.

Theorem 4.3 ([9, Thm. 4.3])**.** *The kernel of Φ is $J(S, S')$. Thus Φ induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^2_{\mathrm{tr}}(S), \mathfrak{h}^2_{\mathrm{tr}}(S')) \cong \frac{\mathrm{CH}^2(S \times S')_{\mathbb{Q}}}{J(S, S')}.$$

Proof. We first show $J(S, S') \subseteq \ker(\Phi)$. Suppose that $f \in \mathrm{CH}^2(S \times S')_{\mathbb{Q}}$ does not dominate S, hence f can be taken to be a cycle with support in $C \times S'$ for some subvariety C of S of dimension at most 1. We may write

$$f = g \circ i^*,$$

where g is a divisor on $C \times S'$ and $i \colon C \to S$ is the inclusion. Replacing C by its normalization, we may assume that C is smooth (of course thereby dropping the assumption that i is a closed immersion). Arguing componentwise, we may assume that C is connected. Since the case of C being a point is easy by dimension considerations, we assume from now on that C is a smooth, connected curve.

To show that $\Phi(f) = p^2_{\mathrm{tr},S'} \circ f \circ p^2_{\mathrm{tr},S}$ vanishes, it is then enough to prove that $p^2_{\mathrm{tr},S'} \circ g = 0$. By Corollary 3.7, and because $p^2_{\mathrm{tr},S'} \circ p^2_{S'} = p^2_{\mathrm{tr},S'}$, we have

$$p^2_{\mathrm{tr},S'} \circ g \circ p^i_C = 0$$

for $i \neq 2$. Thus $p^2_{\mathrm{tr},S'} \circ g = p^2_{\mathrm{tr},S'} \circ g \circ p^2_C$. Apply Lieberman's lemma twice, [17, Lem. 2.1.3.], using that $p^2_C = [C \times Z_0]$ for some effective divisor Z_0 on C, and get

$$p^2_{\mathrm{tr},S'} \circ g \circ p^2_C = p^2_{\mathrm{tr},S'} \circ [C \times g_*(Z_0)] = [C \times (p^2_{\mathrm{tr},S'})_*(g_*(Z_0))].$$

By Remark 2.8, $p^2_{\mathrm{tr},S'}$ acts as zero on divisors. Thus $(p^2_{\mathrm{tr},S'})_*(g_*(Z_0))$ vanishes, thereby proving that $p^2_{\mathrm{tr},S'} \circ g = 0$ and hence $f \in \ker(\Phi)$.

If on the other hand $f \in J(S, S')$ does not dominate S', then arguing as above for f^t yields

$$p^2_{\mathrm{tr},S} \circ f^t \circ p^2_{\mathrm{tr},S'} = 0.$$

Taking transposes shows that $f \in \ker(\Phi)$. This finishes the proof that $J(S, S') \subseteq \ker(\Phi)$.

For the reverse inclusion, take some $f \in \ker(\Phi)$. We base change to $K = K(S)$, the function field of S. Note that by Lemma A.1 (i) the pullbacks of p^i_S and $p^i_{S'}$ still yield a Chow–Künneth decomposition for S_K and S'_K, respectively. We abuse notation, and still write f for the pullback of f to $S_K \times_K S'_K$, etc.

Let ξ be the generic point of S, which we view as zero-cycle on S_K. By Lemma 4.1 above there is a curve C in S and a zero-cycle W with support in C_K such that

$$(p^2_{\mathrm{tr},S})_*(\xi) = [\xi] + W.$$

Apply f_* to both sides and use Lemma 4.1 again to find a curve C' in S' and a zero-cycle W' with support in C'_K such that

$$(p^2_{\mathrm{tr},S'} \circ f \circ p^2_{\mathrm{tr},S})_*(\xi) = f_*(\xi) + f_*(W) + W'.$$

But $p^2_{\mathrm{tr},S'} \circ f \circ p^2_{\mathrm{tr},S} = 0$ by assumption. Thus

$$f_*(\xi) + f_*(W) + W' = 0. \tag{$**$}$$

Now we regard S'_K as $\xi \times S'$ lying inside $S \times S'$, and take Zariski closures. The Zariski closures of $f_*(W)$ and W' have support contained in $S \times f_*(C)$ and $S \times C'$, respectively. Hence, the class of their sum f_1 lies in $J(S, S')$. By equation $(**)$ and Corollary A.3 the sum $f_2 = f + f_1$ does not dominate S, and hence lies in $J(S, S')$. Thus $f = f_2 - f_1 \in J(S, S')$. □

4.2. Let X be a smooth, connected variety of dimension d and let K be a finitely generated field extension of k of transcendence degree t. For $0 \le i \le t$, let $\mathrm{F}^i(X_K) = \mathrm{F}^i \mathrm{CH}_0(X_K)_{\mathbb{Q}}$ be the subspace of $\mathrm{CH}^d(X_K)_{\mathbb{Q}}$ generated by the images of the $\mathrm{CH}^d(X_L)_{\mathbb{Q}}$, where L runs over all intermediate field extension $k \subseteq L \subseteq K$ of transcendence degree at most i over k. This gives a finite, increasing filtration on the rational Chow group of zero-cycles of X_K,

$$\mathrm{F}^0(X_K) \subseteq \mathrm{F}^1(X_K) \subseteq \mathrm{F}^2(X_K) \subseteq \cdots \subseteq \mathrm{F}^t(X_K) = \mathrm{CH}_0(X_K)_{\mathbb{Q}}.$$

It is easy to see that the filtration is natural with respect to taking pullback and push-forward along arbitrary correspondences.

The next result shows that the length of the filtration is bounded by the dimension of X.

Proposition 4.4. $\mathrm{F}^d(X_K) = \mathrm{CH}_0(X_K)_{\mathbb{Q}}$.

Proof. We have to show that for every class $[P]$ of a closed point $P \in X_K$ lies in $\mathrm{F}^d(X_K)$. Let V be a variety over k with function field $K(V) = K$. We can view X_K as $\eta_V \times X$ lying inside $V \times X$, so that we can take $Y \subseteq V \times X$ to be the closure of P, which is an integral subvariety, and let $f = [Y]$ be its class in $\mathrm{CH}^d(V \times X)_{\mathbb{Q}}$. Note that $f_*(\eta_V) = [P]$ by Lemma A.2.

Let p, q be the projections from Y to V and X. Note that Y dominates V, and both have the same dimension, so the morphism p is generically finite. Thus $p^*(\eta_V) = [\eta_Y]$.

We consider the graph Γ_q as a cycle on $Y \times X$. Observe that $Y = (p \times \mathrm{id}_X)_*(\Gamma_q)$ in $V \times X$. By Lieberman's lemma, [17, Lem. 2.1.3.], this gives the identity of correspondences

$$f = q_* \circ p^*.$$

Evaluate this equation at η_V to get $[P] = q_*(\eta_Y)$ in $\mathrm{CH}_0(X_K)_{\mathbb{Q}}$. Let $Q \coloneqq q(\eta_Y) \in X$, and let $L' = \bar{k}(Q)$, the algebraic closure of the residue field of the point $Q \in X$. The transcendence degree of L' over k is at most d. Write $Q_{L'}$ for the closed point of $X_{L'}$ determined by Q.

The morphisms p and q give inclusions of fields $K \subseteq K(Y)$ and $k(Q) \subseteq K(Y)$. Set $L = L' \cap K$, and let $\pi \colon X_K \to X_L$ be the canonical morphism. Note that Q defines a closed point Q_L on X_L, because L'/L is algebraic. Moreover, L is algebraically closed in K, so $\pi^{-1}(Q_L)$ consists of just one point in X_K, namely Q_K. So $[P] = [Q_K] = \pi^*[Q_L]$, and L has transcendence degree at most d. This proves $[P] \in \mathrm{F}^d(X_K)$. □

4.3. We aim for a second description of $\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^2_{\mathrm{tr}}(S), \mathfrak{h}^2_{\mathrm{tr}}(S'))$ for two surfaces S and S'. Let $K = K(S)$ denote the function field of S. It is interesting how the filtration $\mathrm{F}^i \mathrm{CH}_0(S_K)$ studied above relates to $J(S, S')$, the subspace of $\mathrm{CH}^2(S \times S')_{\mathbb{Q}}$ from Section 4.1:

Lemma 4.5 ([9, Lem. 4.7])**.** $J(S,S') = \left\{ f \in \mathrm{CH}^2(S \times S')_{\mathbb{Q}} \mid f_*(\xi) \in \mathrm{F}^1(S_K) \right\}.$

Proof. "$\subseteq$" Suppose that f lies in $J(S,S')$. If f does not dominate S, then $f_*(\xi) = 0$ and there is nothing to prove. Thus we assume that f does not dominate S'. As in the proof of Theorem 4.3 we may and do assume that there is a curve $i\colon C \hookrightarrow S'$ and a correspondence $g \in \mathrm{CH}^1(S \times C)$ such that $f = i_* \circ g$. By Proposition 4.4 we see that $g_*(\xi) \in \mathrm{F}^1(C_K)$, and since the filtration is natural, $f(\xi) = i_*(g_*(\xi))$ lies in $\mathrm{F}^1(S_K)$.

"$\supseteq$" Take a correspondence $f \in \mathrm{CH}^2(S \times S')_{\mathbb{Q}}$ and set $Z = f_*(\xi) \in \mathrm{CH}^2(S'_K)$. Suppose that $Z = \pi^* Z'$ for some cycle Z' on S'_L, where L is an intermediate field $k \subseteq L \subseteq K$ of transcendence degree at most 1 and $\pi\colon S'_K \to S'_L$ is the canonical morphism.

Let C be a curve with function field L. The field extension K/L induces a dominant rational map $r\colon S \dashrightarrow C$. Let $U \subseteq S$ be an open subset where r is defined. Note that r flat, because it is a dominant map from an integral scheme to a smooth curve, see [5, Prop. III.9.7]. We get the following commutative diagram of Chow groups:

$$\begin{array}{ccccc}
\mathrm{CH}^2(U \times S')_{\mathbb{Q}} & \longrightarrow & \mathrm{CH}^2(S'_K)_{\mathbb{Q}} & \qquad & Z \\
\uparrow{\scriptstyle (r\times \mathrm{id})^*} & & \uparrow{\scriptstyle \pi^*} & & \uparrow \\
\mathrm{CH}^2(C \times S')_{\mathbb{Q}} & \longrightarrow & \mathrm{CH}^2(S'_L)_{\mathbb{Q}} & & Z'
\end{array}$$

We write f' for the closure of Z' in $\mathrm{CH}^2(C \times S')$. By dimension reasons, f' does not dominate S'. So, if f_1 is the closure of $(r \times \mathrm{id})^*(f')$ in $S \times S'$, then its class lies in $J(S,S')$. By the commutativity of the diagram above $(f_1)_*(\xi) = Z = f_*(\xi)$. Hence, $f_2 := f - f_1$ does not dominate S, and its class lands in $J(S,S')$. We conclude that $f = f_1 + f_2 \in J(S,S')$. ☐

Consider the map

$$\tilde{\Psi}\colon\ \mathrm{CH}^2(S \times S')_{\mathbb{Q}} \to \mathrm{CH}^2(S'_K)_{\mathbb{Q}} \colon f \mapsto \Phi(f)_*(\xi).$$

Here, $\Phi(f) = p^2_{\mathrm{tr},S'} \circ f \circ p^2_{\mathrm{tr},S}$ as defined in Section 4.1. By Remark 2.8 the map lands in

$$\mathrm{T}(S'_K) := \ker(\mathrm{AJ}\colon\ \mathrm{CH}^2(S'_K)_0 \to \mathrm{Alb}_{S'_K}(K)) \otimes \mathbb{Q},$$

the Albanese kernel with rational coefficients. The filtration of Section 4.2 restricts to a filtration $\mathrm{F}^i\,\mathrm{T}(S'_K)$ on the Albanese kernel.

Theorem 4.6 ([9, Thm. 4.8])**.** *The map*

$$\Psi\colon\ \mathrm{CH}^2(S \times S')_{\mathbb{Q}} \to \frac{\mathrm{T}(S'_K)}{\mathrm{F}^1\,\mathrm{T}(S'_K)} \colon f \mapsto [\Phi(f)_*(\xi)]$$

is surjective and has kernel $J(S,S')$. Thus it induces an isomorphism

$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^2_{\mathrm{tr}}(S), \mathfrak{h}^2_{\mathrm{tr}}(S')) \cong \frac{\mathrm{T}(S'_K)}{\mathrm{F}^1\,\mathrm{T}(S'_K)}.$$

Proof. For the surjectivity, take $Z \in \mathrm{T}(S'_K)$. Let f be its Zariski closure in $S \times S'$. We write $f_1 = \Phi(f)$ and $f_2 = f - f_1$. Since Φ is idempotent, f_2 lies in $\ker(\Phi) = J(S, S')$. Lemma 4.5 implies that $(f_2)_*(\xi)$ lies in $\mathrm{F}^1\,\mathrm{T}(S'_K)$. We conclude that

$$\Psi(f) = (f_1)_*(\xi) = f_*(\xi) - (f_2)_*(\xi) = Z - (f_2)_*(\xi),$$

so that $\Psi(f) = [Z]$.

The inclusion $J(S, S') \subseteq \ker(\Psi)$ follows immediately from the fact that $J(S, S')$ is the kernel of Φ, see Theorem 4.3. On the other hand, if $f \in \ker(\Psi)$, then $\Phi(f)_*(\xi) \in \mathrm{F}^1(S'K)$, so by Lemma 4.5 we get $\Phi(f) \in J(S, S')$, so $\Phi(f) = 0$. This shows $\ker(\Psi) \subseteq J(S, S')$. By Theorem 4.3 this proves the desired isomorphism. □

4.4. We collect some corollaries of the descriptions above. The first one is a reformulation of Question 3.11.

Corollary 4.7. *For surfaces S and S' the following are equivalent:*

(i) $\mathrm{CH}^2(S \times S')_{\mathbb{Q},\mathrm{hom}} \subseteq J(S, S')$.

(ii) $\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^2(S), \mathfrak{h}^2(S')) = \mathrm{Hom}_{\mathcal{M}_{\mathrm{hom}}}(\mathfrak{h}^2(S), \mathfrak{h}^2(S'))$.

(iii) *Conjecture* 3.3 (B) *holds true for* $\mathrm{CH}^2(S \times S')_{\mathbb{Q}}$.

Proof. (i) ⇒ (ii) In the category of Artin motives rational and homological equivalence coincide, so $\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^2_{\mathrm{alg}}(S), \mathfrak{h}^2_{\mathrm{alg}}(S')) = \mathrm{Hom}_{\mathcal{M}_{\mathrm{hom}}}(\mathfrak{h}^2_{\mathrm{alg}}(S), \mathfrak{h}^2_{\mathrm{alg}}(S'))$. By Proposition 2.9 it is enough to show $\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}^2_{\mathrm{tr}}(S), \mathfrak{h}^2_{\mathrm{tr}}(S')) = \mathrm{Hom}_{\mathcal{M}_{\mathrm{hom}}}(\mathfrak{h}^2_{\mathrm{tr}}(S), \mathfrak{h}^2_{\mathrm{tr}}(S'))$. By assumption, if $f \in \mathrm{CH}^2(S \times S')_{\mathbb{Q}}$ is homologically trivial, then it lies in $J(S, S')$. If moreover $f = p^2_{\mathrm{tr},S'} \circ f \circ p^2_{\mathrm{tr},S}$, then it is zero by Theorem 4.3.

(ii) ⇒ (i) Suppose that $f \in \mathrm{CH}^2(S \times S')_{\mathbb{Q},\mathrm{hom}}$. By assumption $p^2_{\mathrm{tr},S'} \circ f \circ p^2_{\mathrm{tr},S} = 0$, and by Theorem 4.3 this implies $f \in J(S, S')$.

(iii) ⇔ (ii) One direction is just Proposition 3.5 (ii). The converse follows by tracking its proof backwards, employing Corollary 3.10. □

Can we characterize geometrically when $\mathfrak{h}^2_{\mathrm{tr}}(S) = 0$? This works best for $k = \mathbb{C}$:

Corollary 4.8. *Let S be a surface with function field $K = K(S)$. The following are equivalent:*

(i) *The transcendental motive $\mathfrak{h}^2_{\mathrm{tr}}(S)$ is zero.*

(ii) *The class of the diagonal $[\Delta_S] \in \mathrm{CH}^2(S \times S)_{\mathbb{Q}}$ is a $\mathbb{Q}$-linear combination of cycle classes not dominating the first or the second factors, i.e. $[\Delta_S] \in J(S, S)$.*

(iii) $\mathrm{T}(S_K) = 0$.

Suppose further that the ground field k is an uncountable algebraically closed field (e.g. $k = \mathbb{C}$). Then conditions (i)-(iii) *are further equivalent to*

(iv) *The Albanese kernel of S is zero.*

Proof. (i) $\Leftrightarrow$ (ii) By Theorem 4.3 there is an isomorphism $\mathrm{CH}^2(S \times S)_{\mathbb{Q}}/J(S,S) \cong \mathrm{End}(\mathfrak{h}^2_{\mathrm{tr}}(S))$ mapping the class of the diagonal to the identity. So $\mathfrak{h}^2_{\mathrm{tr}}(S) = 0$ if and only if $[\Delta_S] \in J(S,S)$.

(i) $\Leftrightarrow$ (iii) Clearly $\mathfrak{h}^2_{\mathrm{tr}}(S) = 0$ implies $\mathfrak{h}^2_{\mathrm{tr}}(S_K) = 0$ and hence $\mathrm{T}(S_K) = 0$, see Remark 2.8. On the other hand, by Theorem 4.6 $\mathrm{T}(S_K) = 0$ implies $\mathrm{End}(\mathfrak{h}^2_{\mathrm{tr}}(S)) = 0$, so $\mathfrak{h}^2_{\mathrm{tr}}(S) = 0$.

(i) $\Rightarrow$ (iv) By Remark 2.8, the Albanese kernel of S vanishes up to torsion, but it is always torsion-free by Rojtman's theorem, [18].

(iv) $\Rightarrow$ (i) Since everything is of finite type over k, there is a finitely generated subfield $k' \subset k$ and a surface S' over k' such that $S = S' \times_{k'} k$. The assumptions on k assures the field $K' = K(S')$ can be abstractly embedded into k. We see that the map $\mathrm{T}(S' \times K') \to \mathrm{T}(S)$ is injective, so that $\mathrm{T}(S'_{K'}) = 0$. By Theorem 4.6, this implies that $\mathfrak{h}^2_{\mathrm{tr}}(S') = 0$, and so $\mathfrak{h}^2_{\mathrm{tr}}(S) = 0$. □

5. Curves on Surfaces

Let S be a surface containing a curve C. We are interested in the image of $\mathrm{CH}_0(C)$ in $\mathrm{CH}_0(S)$. There are two extreme cases:

(i) The induced map $\mathrm{CH}_0(C)_0 \to \mathrm{CH}_0(S)_0$ is zero.

(ii) The induced map $\mathrm{CH}_0(C) \to \mathrm{CH}_0(S)$ is surjective.

In the first case, we call C a *(pointwise) constant cycle curve*. We discuss how this condition can be expressed in terms of the transcendental motive of S in Section 5.1.

The second case is much rarer. In Section 5.2, we will show that for complex surfaces the transcendental motive of S is indeed an obstruction to the existence of such curves, and that if they exist they can always be taken to be smooth hyperplane sections.

5.1. Let $i\colon C \to S$ be a non-constant morphism from a curve to a surface. As the notation suggests, we are mainly interested in the case where i is the inclusion of a non-singular curve on S, or a normalization of a singular curve on S.

Definition 5.1. We call C a *pointwise constant cycle curve* if the induced map on zero-cycles $i_*\colon \mathrm{CH}_0(C)_0 \to \mathrm{CH}_0(S)_0$ is zero.

In other words we require that every point in C defines the same class in S (up to degree). This notion is problematic, for example it is not stable under base change for small fields. A better definition is the following, introduced by Huybrechts, [7]:

Definition 5.2. We call C a *constant cycle curve* if the map $i_*\colon \mathrm{CH}_0(C_K)_{0,\mathbb{Q}} \to \mathrm{CH}_0(S_K)_{0,\mathbb{Q}}$ is zero, where $K = K(C)$ is the function field of C.

For example, rational curves are always constant cycle curves, since $\mathrm{CH}^1(\mathbb{P}^1)_0 = 0$. In [7], Huybrechts studies constant cycle curves on K3 surfaces, and shows that there are many non-rational constant cycle curves. He uses a slightly more technical definition, which works better for K3 surfaces. The main result of this section is a reformulation of our definition in terms of motives.

Theorem 5.3. *A map $i\colon C \to S$ exhibits C as a constant cycle curve if and only if the induced morphism of Chow motives $i_*\colon \mathfrak{h}^1(C)(1) \to \mathfrak{h}^2_{\mathrm{tr}}(S) \oplus \mathfrak{h}^3(S)$ is zero.*

Proof. Assume that C is a constant cycle curve. We view the generic point η_C as a zero-cycle on C_K. Observe that $(p^1_C)_*(\eta_C)$ is numerically trivial, because $\mathrm{CH}^1(\mathfrak{h}^1(C_K))_{\mathbb{Q}} = \mathrm{CH}^1(\mathfrak{h}^1(C_K))_{0,\mathbb{Q}}$. Since C is a constant cycle curve, we get that $(i_* \circ p^1_C)_*(\eta_C) = 0$ in $\mathrm{CH}^2(S_K)$. In other words, $i_* \circ p^1_C$ does not dominate C, hence it can be chosen to be a cycle supported on a set of the form $X \times S$, for some zero-dimensional subset $X \subsetneq C$, see Corollary A.3 in the Appendix. So $i_* \circ p^1_C$ is a linear combination of cycles of the form $[x \times D]$, where D is a divisor on S. However $q \coloneqq p^2_{\mathrm{tr},S} + p^3_S$ acts as zero on divisors, see Section 2.4, so by Lieberman's lemma $q \circ ([x \times D]) = [x \times q_*(D)] = 0$. This shows that $q \circ i_* \circ p^1_C = 0$, which proves one direction.

On the other hand, assume that $i_*\colon \mathfrak{h}^1(C)(1) \to \mathfrak{h}^2_{\mathrm{tr}}(S) \oplus \mathfrak{h}^3(S)$ is zero. Let Z be a numerically trivial divisor on C_K. We find a correspondence $f \in \mathrm{CH}^2(C \times C)$ with $f_*(\eta_C) = Z$, for example the closure of Z in $C \times S$. We may replace f by $p^1_C \circ f$, because p^1_C acts as the identity on numerically trivial divisors. To show that $i_*(Z) = 0$ it is enough to prove that $(i_* \circ f)_*(\eta_C) = 0$.

Write again $q \coloneqq p^2_{\mathrm{tr},S} + p^3_S$. Since $p^2_{\mathrm{tr},S}$ and p^3_S are the only projectors of the refined Chow–Künneth decomposition acting nontrivially on numerically trivial divisors, we have

$$(i_* \circ f)_*(\eta_C) = q_*((i_* \circ f)_*(\eta_C)) = (q \circ i_* \circ f)_*(\eta_C).$$

Now by assumption $q \circ i_* \circ p^1_C = 0$, which together with $p^1_C \circ f = f$ shows that this vanishes. $\square$

From the motivic characterization it is clear that the notion of a constant cycle curve behaves well under extension of scalars and Galois descent. The next result addresses how this is related to pointwise constant cycle curves, see also [7, Prop. 3.7].

Proposition 5.4. *Any constant cycle curve is a pointwise constant cycle curve. If k is uncountable and algebraically closed, then the converse also holds.*

Proof. For the first statement, use Theorem 5.3 and the descriptions of Chow groups of $\mathfrak{h}^2_{\mathrm{tr}}(S)$ and $\mathfrak{h}^3(S)$, see Section 2.4. Indeed, we get that the induced map $i_*\colon \mathrm{CH}_0(C)_0 \to \mathrm{CH}_0(S)_0$ is zero up to torsion. The Albanese kernel is torsion-free, [18], so it is enough to show that the map $\mathrm{Alb}(i)\colon \mathrm{Pic}^0_C \to \mathrm{Alb}_S$ is zero. By Theorem 2.3 it is a torsion element in $\mathrm{Hom}_{\mathrm{AV}}(\mathrm{Pic}^0_C, \mathrm{Alb}_S)$, so it is zero by Lemma 2.4.

For the second statement, note that since k is algebraically closed, the closed points of C_K correspond to closed points of C or to the generic point η_C. By assumption, all closed points of C are mapped to the same class in $[x_0]$ in S. It is enough to show that the generic point is also mapped to $[x_0]$. Indeed, $f \coloneqq i_* - [C \times \{x_0\}] \in \mathrm{CH}^2(C \times S)$ satisfies $f_*(x) = 0$ for all closed points $x \in C$, so that $n \cdot f$ is rationally equivalent to some cycle not dominating C, for some integer $n > 0$, by [21, Cor. 10.20]. This is a variant of the classical Bloch–Srinivas construction from [3]; here the assumption that k is uncountable comes in. By Corollary A.3, we get $n \cdot (i_*(\eta_C) - [x_0]) = nf_*(\eta_C) = 0$ so that in $\mathrm{CH}^2(S_K)_{\mathbb{Q}}$, we get $i_*(\eta_C) = [x_0]$. □

5.2. In this final section we work over the complex numbers to avoid technical complications. The work so far gives a nice result similar to Mumford's famous theorem, [12]. See also the discussion in [21, Ch. 10].

Theorem 5.5. *The Albanese kernel of a complex algebraic surface S vanishes if and only if there exists a closed algebraic subset $C \subsetneq S$ such that $\mathrm{CH}_0(C) \to \mathrm{CH}_0(S)$ is surjective. In this case, C may be taken to be any smooth hyperplane section.*

By Corollary 4.8 the Albanese kernel vanishes if and only if $\mathfrak{h}^2_{\mathrm{tr}}(S) = 0$. In other words, the theorem asserts that $\mathfrak{h}^2_{\mathrm{tr}}(S)$ is the obstruction to S having the Chow group of a curve.

Proof. If there exists such a subset C, then we get a decomposition of a multiple of the diagonal $[\Delta_S]$ by the Bloch–Srinivas construction, [21, Cor. 10.21]. So condition (ii) of Corollary 4.8 is fulfilled, which proves that the Albanese kernel of S vanishes.

Conversely, suppose that the Albanese kernel vanishes. Let C be a smooth hyperplane section, which exists by Bertini's theorem, [5, Thm. II.8.18]. By the weak Lefschetz theorem, [21, Thm. 1.23], the restriction map $i^*\colon H^1(S, \mathbb{C}) \to H^1(C, \mathbb{C})$ is injective, and so by Hodge decomposition, this implies the injectivity of

$$(*) \qquad i^*\colon H^0(S, \Omega^1_S) \to H^0(C, \Omega^1_C).$$

Over the complex numbers, $\mathrm{Alb}_S = H^0(S, \Omega^1_S)^*/H_1(S, \mathbb{Z})$ and $\mathrm{Jac}_C = H^0(C, \Omega^1_C)^*/H_1(C, \mathbb{Z})$, [6, Def. 3.3.7], so the injectivity $(*)$ of implies surjectivity of $\mathrm{Alb}(i)\colon \mathrm{Jac}_C \to \mathrm{Alb}_S$. Since the Albanese kernel is zero, this proves that $i_*\colon \mathrm{CH}_0(C)_0 \to \mathrm{CH}_0(S)_0$ is surjective. Since all closed points on S are numerically equivalent, the assertion follows. □

Recall the following birational invariants of a complex surface S:

$$p_g(S) = \dim H^0(S, \Omega^2_S), \quad q(S) = \dim H^1(S, \mathcal{O}_S)$$

Assume that we are in the situation of Theorem 5.5, i.e. there exists a smooth hyperplane section $C \subset S$ such that $\mathrm{CH}_0(C) \to \mathrm{CH}_0(S)$ is surjective. Then $\mathfrak{h}^2_{\mathrm{tr}}(S) = 0$, so the transcendental cohomology of S vanishes, so $p_g(S) = 0$ by Hodge theory, see e.g. [6, Sect. 3.3].

If we assume moreover that $q(S) = 0$, then $\mathrm{Alb}_S = 0$, so $\mathfrak{h}^3(S) = 0$. It follows by Theorem 5.3 that C is a constant cycle curve. So $\mathrm{CH}_0(S) \simeq \mathbb{Z}$. Conversely, Bloch's conjecture asserts that if $p_g(S) = q(S) = 0$, then the Albanese kernel of S vanishes, [21, Ch. 11]. It remains unsolved for surfaces of general type.

Appendix A. Scalar extension and descent

In certain situations the given varieties do not contain enough cycles for all construction to work (for example over finite fields). In motivic language this means that there is not a sufficient supply of morphisms. In this case an extension of scalars may be helpful. We collect two useful results concerning Chow–Künneth decompositions and scalar extensions.

Proposition A.1. *Suppose that k'/k is a field extension and X is a variety over k. We write $X_{k'}$ for the base change $X \times_k k'$ and $\pi\colon X_{k'} \to X$ for the canonical morphism.*

(i) *If $p_X^0, \dots, p_X^{2d}$ is a Chow–Künneth decomposition for X, then*

$$p^i_{X_{k'}} \coloneqq (\pi \times \pi)^* p^i_X$$

defines a Chow–Künneth decomposition for $X_{k'}$.

ii) *Assume that k'/k is a finite Galois extension. If $p^0_{X_{k'}}, \dots, p^n_{X_{k'}}$ are mutually orthogonal projectors on $X_{k'}$, then there exist unique mutually orthogonal projectors $p^0_X, \dots, p^n_X$ such that*

$$(\pi \times \pi)^* p^i_X = \frac{1}{[k' : k]} \sum_{\sigma \in \mathrm{Gal}(k'/k)} \sigma^* p^i_{X_{k'}}.$$

In particular, every Chow–Künneth decomposition for X' descends to X.

Proof. (i) It is immediate that the $p^i_{X_{k'}}$ are mutually orthogonal projectors. Their sum is the diagonal, because for every variety Y the map $\mathrm{CH}^*(Y)_{\mathbb{Q}} \to \mathrm{CH}^*(Y_{k'})_{\mathbb{Q}}$ is injective, applied to $Y = X \times X$. They lift the Künneth components of the diagonal, because the following diagram commutes:

$$\begin{array}{ccc} \mathrm{CH}^d(X \times X)_{\mathbb{Q}} & \xrightarrow{(\pi\times\pi)^*} & \mathrm{CH}^d(X_{k'} \times_{k'} X_{k'})_{\mathbb{Q}} \\ \Big\downarrow \mathrm{cl} & & \Big\downarrow \mathrm{cl} \\ H^{2d}(X \times X)(d) & \xrightarrow{(\pi\times\pi)^*} & H^{2d}(X_{k'} \times_{k'} X_{k'}). \end{array}$$

(ii) Pick explicit cycles representing the $p^i_{X'_k}$. Then $\sum \sigma^* p^i_{X_{k'}}$ is $\mathrm{Gal}(k'/k)$-invariant on the level of cycles. One computes that $p^i_X \coloneqq \frac{1}{[k':k]^2}(\pi \times \pi)_*(\sum \sigma^* p^i_{X_{k'}})$ does the job. The injectivity of $(\pi \times \pi)^*$ implies that they are orthogonal projectors and unique. The final assertion follows from the commutativity of the diagram above. See also [19, Lemma 1.17]. □

For instance, when constructing certain correspondences one can extend the ground field, so that there exist enough cycles. With these one constructs needed auxiliary correspondences over the extension field and uses descend to get the correspondences back to the original ground field. This is used for example in Section 2.3.

Another application, as in the proof of Theorem 4.3, is to base change to the function field of a variety. Then the generic point gives a zero-cycle, which sometimes allows analyzing the generic behavior of correspondences. In this context the following lemma is useful.

Lemma A.2. *Let X, Y be smooth, connected, projective varieties and $f \in \mathrm{CH}^d(X \times Y)$. Let $K = K(X)$ be the function field of X and f_K be the pullback of f to $(X \times Y)_K$, viewed as a correspondence from X_K to Y_K. We regard the generic point $\xi \in X$ as a zero-cycle on X_K, and regard Y_K as $\xi \times Y$ lying inside $X \times Y$. Then as elements of $\mathrm{CH}^*(Y_K)$*

$$(f_K)_*(\xi) = f|_{\xi \times Y}.$$

Proof. By the moving lemma we may assume that f_K and $\xi \times_K Y_K$ are prime cycles that intersect properly on $X_K \times_K Y_K$. Their set-theoretical intersection is $f|_{\xi \times Y_K}$, and for each of its irreducible components Z the multiplicity is $\chi\left(\mathcal{O}_{f,Z} \otimes^{\mathrm{L}}_{\mathcal{O}_{(X\times Y)_K,Z}} \mathcal{O}_{\xi \times Y_K, Z}\right)$. Since the inclusion $\xi \hookrightarrow X$ is flat, the tensor product is actually underived. So, the multiplicity coincides with the one given by the scheme theoretic pullback $f|_{\xi \times Y} = f \times_{X \times Y} (\xi \times Y)$. □

In the situation of the lemma we often simply write $f_*(\xi)$ for $(f_K)_*(\xi)$. A cycle on $X \times Y$ *dominates* X if there is no closed proper subset $X' \subsetneq X$ such that it has support on $X' \times Y$. We say that a cycle class $f \in \mathrm{CH}^*(X \times Y)$ dominates X if all cycles with class f dominate X.

Corollary A.3. *A correspondence $f \in \mathrm{CH}^*(X \times Y)$ dominates X if and only if $f_*(\xi) \neq 0$ in $\mathrm{CH}^*(Y_K)$.*

Proof. If f is a cycle not dominating $X \times Y$, then its support does not intersect $\xi \times Y$, so by Lemma A.2 above $f_*(\xi) = f|_{\xi \times Y} = 0$. For the converse, recall that $\mathrm{CH}^*(Y_K) \cong \varinjlim \mathrm{CH}^*(U \times Y)$, where the direct limit goes over all non-empty open subsets of X, [2, Lem. 1.I.20]. Thus, if $f_*(\xi) = 0$, then there is some open $U \subseteq X$ with $f|_{U \times Y} = 0$. Let X' be the complement of U. By the localization exact sequence

$$\mathrm{CH}^*(X' \times Y) \to \mathrm{CH}^*(X \times Y) \to \mathrm{CH}^*(U \times Y) \to 0,$$

we see that f lies in the image of $\mathrm{CH}^*(X' \times Y)$, so f does not dominate X. □

Appendix B. Galois representations and Artin motives

The étale cohomology groups $H^*_{\text{ét}}(X \times_k k^s, \mathbb{Q}_\ell)$ are not only $\mathbb{Q}_\ell$-vector spaces, but are indeed Galois representations, that is discrete representations of the absolute Galois group $\mathrm{Gal}(k^s/k)$. This is also reflected by the category of Chow motives: The category of finite dimensional discrete Galois representations embeds into the category of Chow motives, and its image is generated by 0-dimensional motives, so called *Artin motives*. Since there does not seem to be an appropriate exposition on Artin motives, we develop the basic facts here. Note that this appendix is not necessary for readers only interested in separably closed ground fields.

For us, a *Galois representation* of the Galois extension k'/k is a discrete, finite dimensional representation of the Galois group $\mathrm{Gal}(k'/k)$ over $\mathbb{Q}$. If we speak just of a Galois representation, then we mean a Galois representation of the separable closure k^s/k, the field k being implicit.

The representation V being discrete means that the group homomorphism $\mathrm{Gal}(k'/k) \to \mathrm{GL}(V)$ is continuous, where we equip $\mathrm{GL}(V)$ with the discrete and $\mathrm{Gal}(k'/k)$ with the profinite topology. We denote the category of Galois representations of k'/k by $\mathrm{RepG}(k'/k)$. Morphisms are given by $\mathrm{Gal}(k'/k)$ invariant linear maps. The category $\mathrm{RepG}(k'/k)$ is semi-simple abelian.

The following proposition often helps when working with infinite field extensions.

Proposition B.1. *Let V be a Galois representation of an infinite Galois extension K/k. Then there is a finite Galois subextension k'/k such that the action of $\mathrm{Gal}(K/k)$ on V factors through $\mathrm{Gal}(k'/k)$.*

Proof. Let G be the image of the continuous group homomorphism $\rho\colon \mathrm{Gal}(K/k) \to \mathrm{GL}(V)$. Since $\mathrm{Gal}(K/k)$ is profinite, G is compact. But G is also discrete, hence G is finite.

Now $\ker(\rho)$ is a closed normal subgroup of $\mathrm{Gal}(K/k)$, and hence corresponds to a Galois subextension k'/k with $\mathrm{Gal}(K/k') = \ker(\rho)$. However $\mathrm{Gal}(k'/k) \cong \mathrm{Gal}(K/k)/\mathrm{Gal}(K/k') \cong G$ is a finite group, so k'/k has finite degree. □

Consider the full subcategory of $\mathcal{M}_{\mathrm{rat}}$ generated by $\mathfrak{h}(X)$ for X a zero-dimensional, smooth, projective variety, possibly disconnected. We observe that for such X the set $X(k^s)$ of geometric points comes with a natural $\mathrm{Gal}(k^s/k)$-action. (Note that the structure map $X \to \operatorname{Spec} k$ is finite étale and hence separable; thus $X(k^s)$ is indeed the set of geometric points.) Thus $\mathbb{Q}^{X(k^s)}$ is a Galois representation, which we denote by $\mathrm{Rep}(X) := \mathrm{Rep}(\mathfrak{h}(X))$. We denote the basis vector corresponding to $x \in X(k^s)$ by b_x.

Note that Rep is functorial: Suppose X, Y are zero-dimensional smooth, projective varieties and let $f \in \mathrm{CH}^0(X \times Y)_{\mathbb{Q}}$ be a correspondence. By base change we get a cycle $f_{k^s} \in CH^0(X_{k^s} \times_{k^s} Y_{k^s})_{\mathbb{Q}}$. Since all points in $X_{k^s} \times_{k^s} Y_{k^s}$ are k^s-rational, we can write uniquely

$$f_{k^s} = \sum_{x,y} \alpha_{x,y}[x \times y] \text{ for some } \alpha_{x,y} \in \mathbb{Q},$$

where x and y run through $X(k^s)$ and $Y(k^s)$. (Note that since X is smooth, the map $X \to \operatorname{Spec} k$ is separable; so $X(k^s) = X_{k^s}(k^s)$.) We define $\mathrm{Rep}(\gamma)\colon \mathrm{Rep}(X) \to \mathrm{Rep}(Y)$ to be the map sending b_x to $\sum_y \alpha_{x,y} b_y$. Since the cycle f_{k^s} originates from $\mathrm{CH}^0(X \times Y)_{\mathbb{Q}}$, the map is indeed $\mathrm{Gal}(k^s/k)$-equivariant. One easily checks that $\mathrm{Rep}(f \circ g) = \mathrm{Rep}(f) \circ \mathrm{Rep}(g)$.

Let $\mathcal{M}_{\mathrm{Art}}$ be the full subcategory of $\mathcal{M}_{\mathrm{rat}}$ generated by $(X, p, 0)$ for X zero-dimensional and $p \in \mathrm{CH}^0(X \times X)_{\mathbb{Q}}$ any projector. We call those motives *Artin motives*, [1, Ex. 4.1.6.1].

Proposition B.2. *The functor* Rep *extends to an equivalence $\mathcal{M}_{\mathrm{Art}} \to \mathrm{RepG}(k^s/k)$.*

Proof. By definition $\mathcal{M}_{\mathrm{Art}}$ is the pseudo-abelian hull of the category generated by the $\mathfrak{h}(X)$, and $\mathrm{RepG}(k^s/k)$ is abelian. Thus, Rep extends to a functor $\mathcal{M}_{\mathrm{Art}} \to \mathrm{RepG}(k^s/k)$.

To prove that Rep is fully faithful, it is enough to show that

$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(\mathfrak{h}(X), \mathfrak{h}(Y)) \to \mathrm{Hom}_{\mathrm{RepG}(k^s/k)}(\mathrm{Rep}(X), \mathrm{Rep}(Y))$$

is bijective for all 0-dimensional varieties X, Y. Injectivity follows from the injectivity of the base change map $\mathrm{CH}^0(X \times Y)_{\mathbb{Q}} \to \mathrm{CH}^0((X \times Y)_{k^s})_{\mathbb{Q}}$. For surjectivity we note that any linear map $\phi\colon \mathrm{Rep}(X) \to \mathrm{Rep}(Y)$ is represented by a matrix $A = (\alpha_{x,y})$ such that $\phi(b_x) = \sum_y \alpha_{x,y} b_y$. Because A commutes with the Galois action, the cycle $\sum \alpha_{x,y}[x \times y]$ descends to $\mathrm{CH}^0(X \times Y)_{\mathbb{Q}}$, which is a preimage for ϕ.

It remains to prove that Rep is essentially surjective. Let V be a Galois representation. By Proposition B.1, V is actually a representation of $G = \mathrm{Gal}(k'/k)$ for some finite Galois extension k'/k. By Maschke's theorem we may assume that V is irreducible. Then V is a direct summand of the regular representation $\mathbb{Q}[G]$, [4, Cor. 2.18]. However, as $\mathrm{Gal}(k^s/k)$-sets, G and $\mathrm{Spec}(k')(k^s) = \mathrm{Hom}_k(k', k^s)$ are isomorphic. Thus the regular representation is $\mathrm{Rep}(\mathrm{Spec}(k'))$. Since Rep is fully faithful and $\mathcal{M}_{\mathrm{Art}}$ is pseudo-abelian, V lies in the essential image of Rep. □

Hence, for every Galois representation V we find a unique Artin motive M such that $\mathrm{Rep}(M) = V$, the *Artin motive associated to* V. We denote it by $\underline{V} = M$. By construction all its cohomology groups vanish except for $H^0(\underline{V})$. For ℓ-adic cohomology, we get $H^0(\underline{V}) \cong V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ as Galois representations (over $\mathbb{Q}_\ell$). Similar results hold for the other classical Weil cohomology theories, for example for Betti cohomology we get $H^0(\underline{V}) \cong V$ as $\mathbb{Q}$-vector spaces. The Chow groups of $\underline{V}$ are computed in the following proposition:

Proposition B.3. *All Chow groups of $\underline{V}$ vanish, except*

$$\mathrm{CH}^0(\underline{V}) = V^{\mathrm{Gal}(k^s/k)}.$$

Proof. Since the varieties underlying Artin motives are zero-dimensional, all Chow groups of higher degree vanish. Again we take a finite Galois extension k'/k such that V is a $G = \mathrm{Gal}(k'/k)$-representation. Then it is enough to prove the assertion for the regular representation $\mathbb{Q}[G]$ (because both sides are additive in V). Nevertheless the Artin motive associated to the regular representation is just $\mathfrak{h}(\mathrm{Spec}(k'))$. Clearly $\mathrm{CH}^0(\mathrm{Spec}(k'))_{\mathbb{Q}} \cong \mathbb{Q}$, and it is classical that the invariant vectors of the regular representation form a one-dimensional subspace. □

References

[1] Y. André. *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, volume 17 of *Panoramas et Synthèses*. SMF Paris, 2004.

[2] S. Bloch. *Lectures on algebraic cycles*. Duke University Mathematics Series, IV. 1980.

[3] S. Bloch and V. Srinivas. Remarks on correspondences and algebraic cycles. *Amer. J. Math.*, 105(5):1235–1253, 1983.

[4] W. Fulton and J. Harris. *Representation theory. A first course*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[5] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Heidelberg, 1977.

[6] D. Huybrechts. *Complex Geometry*. Universitext. Springer-Verlag, Berlin, 2005.

[7] D. Huybrechts. Curves and cycles on K3 surfaces. *Algebraic Geometry 1*, (1):69–106, 2014.

[8] U. Jannsen. Motivic sheaves and filtrations on Chow groups. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 245–302. AMS, Providence, RI, 1994.

[9] B. Kahn, J. P. Murre, and C. Pedrini. On the transcendental part of the motive of a surface. *London Mathematical Society Lecture Note Series*, 1(344):143–202, 2007.

[10] K. Kimura. Murre's conjectures for certain product varieties. *J. Math. Kyoto Univ.*, 47(3):621–629, 2007.

[11] T. Matsusaka. The criteria for algebraic equivalence and the torsion group. *American Journal of Mathematics*, 79(1):53–66, 1957.

[12] D. Mumford. Rational equivalence of 0-cycles on surfaces. *J. Math. Kyoto Univ.*, 9:195–204, 1968.

[13] D. Mumford. *Abelian varieties*, volume 5 of *Tata Inst. Fund. Res. Stud. Math.* Oxford University Press, USA, 1974.

[14] J. P. Murre. On the motive of an algebraic surface. *J. reine angew. Math*, 409(190-204):249, 1990.

[15] J. P. Murre. On a conjectural filtration on the Chow groups of an algebraic variety. I. The general conjectures and some examples. *Indag. Math. (N.S.)*, 4(2):177–188, 1993.

[16] J. P. Murre. On a conjectural filtration on the Chow groups of an algebraic variety. II. Verification of the conjecture for threefolds which are the product of a surface and a curve. *Indag. Math. (N.S.)*, 4(2):189–201, 1993.

[17] J. P. Murre, J. Nagel, and C. Peters. *Lectures on the theory of pure motives*, volume 61 of *University Lecture Series*. AMS, 2013.

[18] A. A. Rojtman. The torsion of the group of 0-cycles modulo rational equivalence. *Ann. of Math. (2)*, 111(3):553–569, 1980.

[19] A. J. Scholl. Classical motives. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 163–187. AMS, Providence, RI, 1994.

[20] C. Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, english edition, 2007. Translated from the French by Leila Schneps.

[21] C. Voisin. *Hodge theory and complex algebraic geometry. II*, volume 77 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, english edition, 2007. Translated from the French by Leila Schneps.